KINEMATIC SYNTHESIS AND ANALYSIS
OF
THREE-LINK SPATIAL FUNCTION GENERATORS WITH HIGHER PAIRS

BY

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With love to

Marita and Noel

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KINEMATIC SYNTHESIS AND ANALYSIS OF THREE-LINK SPATIAL FUNCTION GENERATORS WITH HIGHER PAIRS

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Major Department: Mechanical Engineering

Function generation synthesis of spatial mechanisms with only three links is achieved by employing higher pairs (sphere-plane (Sp), cylinder-plane (Cp) and sphere-groove (Sg) pairs) to constrain the motion of two links.

This dissertation shows the methods and procedures for obtaining the equations for multiply-separated-precision point (MSP) synthesis for four spatial function generators - R-Sp-R, R-Sp-P, R-Cp-C and R-Sg-C.

Higher pair constraint equations in vector form are utilized to obtain closed-form solutions for the different synthesis cases of various numbers of positions and specified and unknown parameters. The method of elimination is used extensively to solve the resulting non-linear systems of equations.

Kinematic synthesis and analysis of the four spatial function generators is performed in vector notations and with screw displacements in vector form. Explicit equations are also obtained from the investigation of the transmission characteristics of these mechanisms.

The synthesis procedures just completed were then augmented by developing design criteria for the pair elements to assure range of motion and avoidance of interference.

CHAPTER 1

INTRODUCTION

1.1 Brief History

Although mechanisms have been designed and used for many centuries, it is only in the recent past that mechanisms have been studied as a branch of science itself. Many theories in mechanisms have been known since the first decades of the 19th century, but most applications were hampered by the limitations of the graphical methods that were generally used. With the introduction of the digital computer, more effort and attention were devoted to the study and advancement of mechanisms. Only since the advent of the digital computer has there been a powerful tool that could be used to carry out, with accuracy and speed, the many tedious and repetitive calculations required in the study of mechanisms.

The subject of mechanisms has a broad coverage. The study can be classified according to the number of links, function of the mechanism, whether it is a planar or a spatial mechanism, whether rigid or elastic links are considered, and so on. However, the two most general classes of mechanisms are the planar mechanism and the spatial mechanism. The study of mechanisms have been classically divided into either analysis of mechanisms or synthesis of mechanisms.

Analysis deals with the study of the motion of an existing mechanism. This involves the determination of displacements, velocities, acceleration and even higher order properties. Analysis also covers topics like transmission characteristics, motion limits and error analysis.

The dynamics of the mechanism are also analyzed and this includes forces, torques, shaking forces, shaking torques and balancing.

Synthesis, on the other hand, deals with the problem of designing mechanisms to fulfill a required motion or task. According to the kind of motion required, the synthesis of mechanisms is classified into three types, namely function generation, path generation and motion generation. Function generation synthesis involves the design of mechanisms which will transmit the motion of one link to another link according to some functional relationship. Path generation synthesis is the process of designing a mechanism wherein one point will describe a desired path. Motion generation synthesis, on the other hand, studies the design of mechanisms that will guide a body through a series of positions and orientations in a plane (in the case of planar mechanisms) or in space (for spatial mechanisms). Motion generation synthesis is also referred to as rigid body guidance.

1.2 Literature Survey

The following literature survey is by no means complete or even extensive. It will present some important and significant works in mechanisms that led to the subject of this work--kinematic synthesis and analysis of three-link spatial function generators. The survey will cite and briefly describe references that have provided important concepts and ideas. Other references not directly related are also covered to show some of the developments and trends in the area of synthesis and analysis of mechanisms.

As mentioned earlier, the study of mechanisms started centuries ago and will surely extend into the future. Hartenberg and Denavit [1] provide a brief and concise review of significant studies in mechanisms before the year 1900. Freudenstein and Sandor [2] presented an extensive review of the different methods, studies and research works in kinematics of mechanisms up to the early part of the 1960's.

Before the advent of the digital computer in the early 1950's, most of the work on mechanisms was on the application of graphical methods to study planar mechanisms. With the availability of the computer, succeeding research and work focused on the use and application of analytical methods. The time consuming, laborious graphical constructions and the limited accuracy of the graphical methods were replaced by the faster and more accurate results obtained by finding the solutions to governing analytical equations.

Several methods have been used and developed for both analysis and synthesis of mechanisms. Algebraic methods using displacement equations were used by Hartenberg and Denavit [1], and Chace [3] used vector methods showing the advantages of using vector notations and operations to obtain closed-form solutions. Sandor [4] in his doctoral dissertation developed a general method of using complex numbers in the synthesis of planar mechanisms. Matrix methods applied to the analysis and synthesis of both planar and spatial mechanisms were also used by the following authors: Denavit [5], Uicker [6], Suh and Radcliff [7], and Suh [8]. Yang [9] utilized quaternion algebra and dual numbers in the analysis of mechanisms. Sandor [10] applied quaternion algebra to the synthesis of spatial mechanisms and Sandor and Bisshopp [11] developed a synthesis method using a stretch-rotation tensor.

It was Freudenstein [12] in his doctoral dissertation who initiated a surge of interest in the synthesis of mechanism. His pioneering work introduced the concept of approximate synthesis of mechanisms --synthesis of mechanisms to exactly satisfy design requirements at the so-called precision points. This led to more research by others to refine and optimize this synthesis procedure. Between the precision points, the mechanism motion will in general deviate from the design requirements. The presence of this deviation, called structural error, led to works by Freudenstein [13], Rose and Sandor [14] and McLarnan [15] which focused on proper spacing of the precision points to reduce the errors. Roth, Freudenstein and Sandor [16] also developed a method of replacing a precision point condition with a transmission angle constraint. Subsequent works in optimization used different techniques to minimize these structural errors.

The mechanism to be synthesized was also subjected to further constraints like link length ratios, transmission angles, mechanism size and even dynamic constraints. A few of these studies were those of Kramer and Sandor [17], Eschenbach and Tesar [18], Garrett and Hall [19] and Sandor and Wilt [20]. Another important synthesis technique is the least squares method. This method sets the criterion that although exactness is not required at the precision points, the sum of the squares of the errors at the precision points must be a minimum. The works of Levitskii and Sarkisyan [21] and Sutherland and Roth [22] are among them.

Other important contributions in synthesis were those of Chen and Roth [23] and Tesar [24] wherein they generalized the concept of precision point synthesis for finitely separated positions (FSP), infinitesimally separated positions (ISP) and combinations of these two. Combination of FSP and ISP, called multiply separated positions (MSP) synthesis, is based on the interchangeability of finite displacement precision points with higher order properties as part of the synthesis specifications.

In recent years, there has been a growing emphasis on developments in the analysis and synthesis of spatial mechanisms. Most of the work in spatial mechanisms has involved mechanisms with lower pairs such as revolute (R), prismatic (P), cylindric (C) and the spheric (S) pairs. Lower pairs are defined as those that have surface contact between elements while higher pairs have line or point contact. Different methods have been used in their analysis and synthesis. Important works in the analysis of spatial mechanisms are those of Chace [25] for vector methods, Duffy [26] for spherical trigonometry, Suh and Radcliff [27] for matrix methods, Yang and Freudenstein [28] for dual numbers and quaternion algebra and Kohli and Soni [29] for successive screw displacements in vector form. Most of these methods were also extended to the synthesis of spatial mechanisms.

The general methods of synthesis for planar mechanisms like approximate synthesis, least squares method and multiply separated precision point synthesis were also applied to the synthesis of spatial mechanisms. Using the spatial four-link RSSR mechanism, Novodrovskii [30] worked on the synthesis problem of function generation with the axes of rotations of the input and output links mutually perpendicular. Stepanoff [31] worked the general problem of having non-intersecting, non-parallel input and output axes. With the RSSR mechanism, Levitskii and Shakvasian [32] applied the least squares technique for finite position synthesis to evaluate up to eight parameters. Rao and Sandor [33, 34] and Rao, Sandor, Kohli and Soni [35] in a series of papers carried out synthesis of four-link mechanisms up to the maximum number of available design parameters. Closed-form solutions were obtained by using the principles of linear superposition on the resulting sets of non-linear synthesis equations. Sandor, Kohli, Reinholtz and Ghosal [36] developed a closed-form analytic solution for a five-link motion generator. Kohli and Soni [37] used successive screw displacements and the concept of pair geometry constraint equations for analysis and synthesis of spatial mechanisms using lower pairs. More recently, Kohli and Singh [38] formulated pair geometry constraint equations for a group of higher pairs. Examples of these were the sphere-plane, the sphere-groove, the slotted-cylinder and sphere,and cylinder-plane pairs. Although various combinations of the S, C, R and P pairs can be used to represent these pairs as in the work of Chen and Roth [39], this group of higher pairs brought to light the apllicability of a single higher pair to duplicate a series or combination of S, C, R or P pairs. Litvin and Gutman [40] used matrix methods for a three-link mechanism with an intermediate higher pair of two crossed cylinders. Sandor, Kohli, Hernandez and Ghosal [41] applied the higher pair geometry constraint equations to analyze three-link spatial mechanisms that contained sphere-plane (Sp) and sphere-groove (Sg) pairs

These last few references provided the impetus for this thesis. It was recognized that with the use of higher pairs, a spatial function generator with only three links can be synthesized. The simplicity of using direct pair geometry constraint equations together with screw displacements in vector form and derivations using vector operations are the main features of this undertaking.

1.3 Objectives

Most, if not all the function generators that have been studied had at least four links. This is true for both planar and spatial mechanisms.

The goal of this dissertation is to synthesize three-link function generators for finitely separated positions (FSP), infinitesimally separated positions (ISP) or multiply separated positions (MSP) specifications. These three-link function generators, made up of only two moving links and a higher pair, possess different numbers of parameters available for synthesis. Several cases of different numbers of specified precision points and sets of specified and unknown parameters will be considered. The synthesized mechanisms will be studied and analyzed for displacements, velocities and accelerations together with motion properties such as transmission characteristics and motion limitations.

Finally, the synthesis of the mechanism will be completed by determining the actual shape and dimensions of the pair elements.

1.4 Summary

The synthesis and analysis of the three-link spatial function generators are carried out in the succeeding chapters. A discussion of lower and higher pairs with their constraints, together with some mathematical background, is covered in Chapter 2. Chapter 3 presents the three-link function generators that will be synthesized. The geometry and equations of motions are also derived. Chapter 4 presents the derivation and results of the MSP synthesis equations for the mechanism considered. Chapter 5 will show the solutions to the different synthesis cases and Chapter 6 will cover the analysis of the spatial function generators. Chapter 7 shows the method for determining the shape and dimensions of the pair elements. Numerical examples are then presented in Chapter 8. Areas of further study and research that will enhance the usefulness of the three-link spatial function generators are discussed in Chapter 9. Finally, the Appendix shows relevant mathematical methods and lists the explicit expressions for the different terms used in the equations and derivations.

CHAPTER 2

REVIEW OF RELATED CONCEPTS

2.1 Lower and Higher Pairs in Spatial Mechanisms

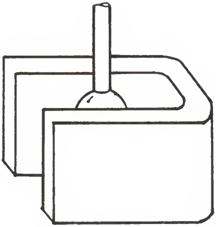
As mentioned in the literature survey, most of the studies in spatial mechanisms considered only lower pairs like the revolute (R), the prismatic (P), the cylinder (C) and the spheric (S) pairs. To satisfy the need for more complex motions, the authors then considered these mechanisms with lower pairs but with increased number of links. This resulted in their studies' becoming more complex.

A common task in the study of spatial mechanisms is the determination of the input/output motion relations between the driving link(s) and the driven link(s). Although this task has been achieved with great success for different mechanisms with many links, the resulting relationships are characterized by very high degree equations. This is due to the higher number of links, and also to the procedure of modeling multi-degrees of freedom pairs into combinations of revolute and prismatic pairs. The expressions derived are series of sequential relative motions between consecutive links in the mechanism. Intermediate link motion variables must then be eliminated so that only input and output motion variables are contained in the final relationship.

Until recently, the common practice in the synthesis of spatial mechanisms was to use dyads or even triads. These are combinations of the four principal pairs R, P, C and S for which generalized expressions have been derived for their applications. Thus, any desired multi-degrees of freedom pair can be obtained by using the proper combinations of these principal pairs.

Looking at the situation from a different point of view, a pair with only two elements can also be devised to duplicate a certain combination or chain of these lower pairs. Singh and Kohli [38] developed some higher and lower pairs and, together with the other commonly used lower pairs, formulated their respective pair geometry constraint equations. Figures 2.1 to 2.4 show some of these pairs with their degrees of freedom, the nature of the relative motion and the pair constraints. These pairs, when represented by combinations of prismatic and revolute pairs, occur with special axis locations and orientations. It is therefore possible to formulate direct constraints for the higher pairs themselves. This simplifies the analysis and synthesis of the mechanisms containing these pairs.

The use of this group of pairs together with the direct constraint equations has unique advantages. Foremost of these is the reduced number of links in the mechanism. Figure 2.5 illustrates this advantage by showing a combination of lower pairs that is equivalent to the cylinder-plane pair. Using the direct pair constraint equations in vectorial form will also result in shorter derivations and more compact expressions. A certain degree of visualization is also possible with the vector expressions. With fewer links in the mechanism, there will also be fewer sources of manufacturing and fabrication errors. This also brings up another advantage, which is the ease of fabricating these pairs.



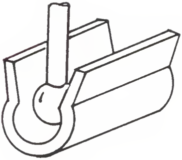
Relative Motions:

Three Rotations
Two Translations

Constraint:

Sphere is constrained to move within the planes

Figure 2.1 Five-DOF pair: sphere-plane pair (Sp)



Relative Motions:

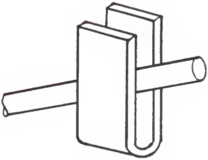
Three Rotations

One Translation

Constraint:

Sphere slides along the groove

Sphere-groove pair (Sg)



Relative Motions:

Two Rotations

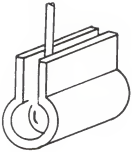
Two Translations

Constraint:

Cylinder motion and its axis is parallel to plane

Cylinder-plane pair (Cp)

Figure 2.2 Four-DOF pairs: cylinder-plane and sphere-groove



Sphere-in-slotted cylinder

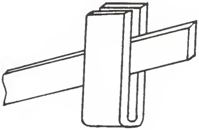
Relative Motions:

Two Rotations

One Translation

Constraint:

- -Sphere cannot rotate about groove axis
- -Sphere slides along the groove



Plane pair

Relative Motions:

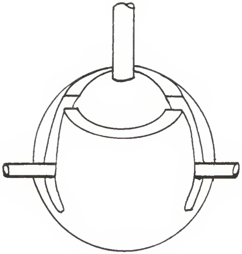
One Rotation

Two Translations

Constraint:

- -Planes can rotate about an axis ⊥ to planes
- -Relative translation is along the plane

Figure 2.3 Three-DOF pairs: sphere-in-slotted cylinder and plane pair



Relative Motions:

Two Rotations

Constraint:

No translations except the the rotation about two ⊥ axis

Figure 2.4 Slotted-spheric pair

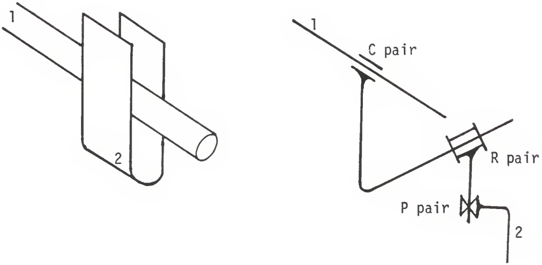


Figure 2.5 Combination of lower pairs equivalent to a Cp pair

2.2 Background Mathematics

The synthesis of the three-link function generators requires some background in certain mathematical areas. Geometry together with vector operations is required and so are methods of solving systems of equations (both linear and non-linear). Knowledge of operations with matrices and determinants is also necessary. In addition, some properties and theorems of matrices and determinants are particularly helpful in the synthesis procedure. Finally, root finding methods for polynomials are required. The following sections will review some of these necessary topics. An extensive coverage of these areas can be found in references [42] and [43].

2.2.1 Solutions to Systems of Non-linear Equations

A common analytical method in solving systems of non-linear equations is by the elimination methods. Non-linearity in the systems of equations encountered exists in the products of the unknown variables. Also, the systems of equations always have an equal number of unknowns and equations.

Two examples will be illustrated to show the method of elimination.

Example 1: Three non-linear equations in three unknowns:

$$a_j x + b_j y + c_j z + d_j xy + e_j xz + f_j = 0 \qquad (2.1)$$

$$j = 1, 2, 3$$

The equations are first arranged to make the coefficients of the variables and the constant term functions of one variable, in this case x.

$$(A_j + B_jx)y + (C_j + D_jx)z + (E_j + F_jx) = 0 \qquad (2.2)$$

$$j = 1, 2, 3$$

Equation (2.2) is now considered as a homogeneous system of three equations in the three unknowns y, z and 1 (for the constant term). As in the case of homogeneous linear equations, the determinant of the coefficient matrix must be zero to have non-trivial solutions for y, z and 1.

This determinant,

$$\begin{vmatrix} (A_1 + B_1x) & (C_1 + D_1x) & (E_1 + F_1x) \\ (A_2 + B_2x) & (C_2 + D_2x) & (E_2 + F_2x) \\ (A_3 + B_3x) & (C_3 + D_3x) & (E_3 + F_3x) \end{vmatrix} = 0 \qquad (2.3)$$

is called the eliminant polynomial - whereby y and z were eliminated from the system of equations and now x is forced to take on values such that equation (2.3) will be true. The determinant of (2.3) will result in a cubic equation in x. Its roots are the solutions for x. To find the solutions for y and z, these values of the roots of x are substituted in any two equations of (2.2) giving a linear system of two equations in two unknowns.

Example 2: Three non-linear equations in three unknowns with additional product terms.

$$a_jx + b_jy + c_jz + d_jxy + e_jxz + f_jyz + g_j = 0 \qquad (2.4)$$

$$j = 1, 2, 3$$

One variable (again x) is chosen as the eliminant variable and equation (2.4) is rearranged as

$$(A_j + B_jx)1 + (C_j + D_jx)z + (E_j + F_jx)y + G_jyz = 0 \qquad (2.5)$$

$$j = 1, 2, 3$$

The three equations of (2.5) can be considered as having four unknown variables 1, z, y and yz. Multiplying the three equations of (2.4) by y will give three more equations in the system and also introduce two additional unknowns, y^2 and y^2z. This makes the number of unknowns and equations equal. The system of six equations in the six unknowns 1, z, y, yz, y^2 and y^2z is now written as

$$\begin{bmatrix} (A_1 + B_1x) & (C_1 + D_1x) & (E_1 + F_1x) & G_1 & 0 & 0 \\ (A_2 + B_2x) & (C_2 + D_2x) & (E_2 + F_2x) & G_2 & 0 & 0 \\ (A_3 + B_3x) & (C_3 + D_3x) & (E_3 + F_3x) & G_3 & 0 & 0 \\ 0 & 0 & (A_1 + B_1x) & (C_1 + D_1x) & (E_1 + F_1x) & G_1 \\ 0 & 0 & (A_2 + B_2x) & (C_2 + D_2x) & (E_2 + F_2x) & G_2 \\ 0 & 0 & (A_3 + B_3x) & (C_3 + D_3x) & (E_3 + F_3x) & G_3 \end{bmatrix} \begin{Bmatrix} 1 \\ z \\ y \\ yz \\ y^2 \\ y^2z \end{Bmatrix} = 0 \qquad (2.6)$$

The method shown in obtaining the coefficient matrix is called Sylvester's Dyalitic Method of Elimination. Other methods exist but Sylvester's method is straightforward and applicable to most forms of non-linear equations.

The determinant of the coefficient matrix is now set to zero to get the eliminant polynomial which expresses the condition for values of x such that the equations will have common solutions.

The roots of x are then substituted back into any five equations of (2.6) to get the values of z, y, yz, y^2 and y^2z. This will result in a linear system of five equations in five unknowns. As a check, the answers obtained must be compatible with the expressions of the unknowns themselves, e.g., the solution for yz must, in fact, be equal to the product of the solutions for y and z.

2.2.2 Evaluation of Determinants

The common methods of evaluating determinants are applicable to the problems in this work but they require tedious expansions and calculations. The matrices encountered have the form shown in equations (2.3) and (2.6). However, in some cases, the matrices are of order six.

The main objective in the evaluation of the determinant is to obtain the polynomial from the determinant since the elements of the matrices are functions of a variable.

Some techniques and methods based on fundamental properties and theories in matrices are illustrated in Appendix A. An indirect method for determining the polynomial will be shown here.*

The indirect method of obtaining the determinant polynomial is achieved by putting in convenient values for the variable in the matrix. This would result in a system of linear equations where the coefficients in the polynomial will be the unknowns. For this method to be exact, the order of the resulting polynomial must be known. (Methods for determining the order of the resulting polynomials are also shown in the appendix). An example with a small matrix will be shown to illustrate the procedure.

---

*Based on a method devised by Dr. Gary Matthew.

Let the following determinant represent the polynomial:

$$|(A_j + B_jx) \quad (C_j + D_jx) \quad (E_j + F_jx)| = P(x) \qquad (2.7)$$

$$j = 1, 2, 3$$

The degree of this polynomial is determined to be cubic so that

$$|(A_j + B_jx) \quad (C_j + D_jx) \quad (E_j + F_jx)| = a_0 + a_1x + a_2x^2 + a_3x^3 \qquad (2.8)$$

$$j = 1, 2, 3$$

Four convenient numerical values are substituted for x in equation (2.8). This will result in the following system of four linear equations in the unknowns a_0, a_1, a_2 and a_3:

$$[\ 1 \quad (x_i) \quad (x_i)^2 \quad (x_i)^3\ ] \quad \{\ a_i\ \} =$$

$$\{\ |\ (A_j + B_jx_i) \quad (C_j + D_jx_i) \quad (E_j + F_jx_i)\ |\ \} \qquad (2.9)$$

$$j = 1, 2, 3$$

$$i = 0, 1, 2, 3$$

where x_i's are the numerical values substituted for x.

CHAPTER 3

THREE-LINK SPATIAL FUNCTION GENERATORS

The three-link spatial mechanisms suitable for function generation will be developed and studied in this chapter. Their geometry will be presented and their equations of motion will then be derived. These mechanisms have the minimum number of three links only because higher pairs are used. Although many three-link mechanisms can be formulated, the mechanisms to be considered are single degree-of-freedom (DOF) mechanisms with one output motion (rotation or translation), or two output motions (rotation and translation) from one input rotation.

3.1 The Degree of Freedom Equation

The number of degrees of freedom of a closed-loop spatial mechanism can be obtained from the following Kutzbach equation:

$$\mathrm{DOF} = 6(n - 1) - 5(P_1) - 4(P_2) - 3(P_3) - 2(P_4) - (P_5) \qquad (3.1)$$

where n = total number of links

P_i = number of pairs in the mechanism with i degrees of freedom

This equation is obtained from the fact that an n-link mechanism will have $(n - 1)$ moving links (with respect to one fixed link) with a general six degrees of freedom motion, and that each of the P_i kinematic pairs of i degrees of freedom will restrict the motion of the interconnected links by $(6 - i)$ degrees of freedom.

For the case of a single DOF mechanism, n = 3 links and either single or dual output motions, equation (3.1) shows that a five DOF pair is required for a single output - single input, one-DOF, three-link mechanism. Also, a four-DOF pair is required for a dual output - single input one-DOF, link mechanism.

Single output - single input

$$a)\quad 1 = 6(3 - 1) - 5(2) - 1(1) \qquad (3.2)$$

Single input - dual output

$$b)\quad 1 = 6(3 - 1) - 5(1) - 4(2) - 2(1) \qquad (3.3)$$

Therefore, the three-link mechanisms to be considered for function generation are of the P_1-P_5-P_1 and P_1-P_4-P_2 types. From the list of lower and higher pairs discussed in Chapter 2, the only five DOF pair is the sphere-plane pair (Sp). The cylinder-cylinder pair proposed by Litvin and Gutnam [40] is also a five-degree-of-freedom pair. However, this pair has to be force-closed whereas the sphere-plane considered has body closure. The two four-DOF pairs that can be used are the sphere-groove (Sg) and the cylinder-plane (Cp) pairs. These, along with the cylindric pair as the two-DOF pair and the revolute and prismatic pairs as the one-DOF are used to obtain the following three-link spatial mechanisms:

A	R-Sp-R
B	R-Sp-P
C	R-Cp-C
D	R-Sg-C

With these mechanisms then, a rotational motion can be coordinated with another rotational motion (R-Sp-R), a rotational motion can be made to produce a coordinated translational motion (R-Sp-P), and a rotational motion can be made to produce a combined rotational and translational motion (R-Cp-C) and (R-Sg-C).

Since the R and P pairs can be derived from the C pair, the R-Sp-C mechanism will be considered for the derivation and then later made into R-Sp-R or R-Sp-P by letting rotations be zero to make a P pair and letting translations be zero to make an R pair.

3.2 Geometry of the Three-Link Function Generators

Figures 3.1 to 3.4 show the R-Sp-R, R-Sp-P, R-Sg-C and R-Cp-P mechanisms generally located in space. Figure 3.5 to 3.8 show the schematic of these mechanisms with the vectors and scalars describing these mechanisms, which are appropriately and conveniently located and oriented with respect to the fixed coordinate system XYZ. The location and orientation of these mechanisms is dictated by the need to simplify the calculations. Also, the number of independent parameters that describe the mechanism can be determined. Thus, the X-axis of the coordinate system is used as the rotation axis for the input revolute pair and the axis of the grounded pair of the second link is defined by a skew angle θ_S measured about the common perpendicular of these two pair axes.

In figure 3.5, the R-Sp-C mechanism is shown with the following vectors and scalars described:

$\underset{\sim}{u}_A$ - unit vector defining the direction of the axis of input link 1 at the revolute pair A. This axis is made to coincide with the X-axis such that $\underset{\sim}{u}_A = 1\hat{i}$

$\underset{\sim}{u}_B$ - unit vector defining the axis of the cylindric pair B. It is defined by the skew angle θ_s measured about the common perpendicular of $\underset{\sim}{u}_A$ and $\underset{\sim}{u}_B$, which is the Y-axis. Thus, $\underset{\sim}{u}_B = \cos\theta_s\hat{i} - \sin\theta_s\hat{k}$.

$\underset{\sim}{Q}$ - vector along the Y-axis, the common perpendicular of $\underset{\sim}{u}_A$ and $\underset{\sim}{u}_B$.

$\underset{\sim}{R}$ - vector from the origin to the point R, the sphere center. This point is fixed to link 1.

$\underset{\sim}{R}'$ - vector from the origin to the point R', a point fixed to link 2 and initially coincident with R. At the starting position then, $\underset{\sim}{R} = \underset{\sim}{R}'$.

$\underset{\sim}{A}$ - a unit vector fixed to link 2 and perpendicular to the plane in the Sp pair.

θ - rotation of link 1 about $\underset{\sim}{u}_A$.

ϕ - rotation of link 2 about $\underset{\sim}{u}_B$.

S_c - linear translation of link 2 along $\underset{\sim}{u}_B$.

Figure 3.5a shows the R-Sp-P mechanism derived from the R-Sp-C of figure 3.5 with the vector $\underset{\sim}{R}$ along the Y-axis and the unit vector $\underset{\sim}{A}$ defining the orientation of the plane. The breakdown of the R-Sp-P into the configuration of figure 3.5a is possible because link 2 does not rotate--making any axis parallel to $\underset{\sim}{u}_B$ possible as the actual prismatic axis. Thus, the coordinate system can be rotated about the X-axis to let $\underset{\sim}{R}$ be along the Y-axis.

Figure 3.6 shows the R-Cp-C mechanism with the vectors and the scalars defining it. The descriptions of the vectors and scalars are the same as in the R-Sp-C mechanism except that the point R is not a sphere center but a point located along the axis of the cylinder. However,

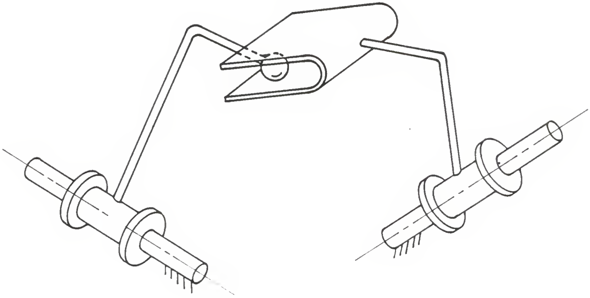


Figure 3.1 The revolute--sphere-plane--revolute (R-Sp-R) mechanism

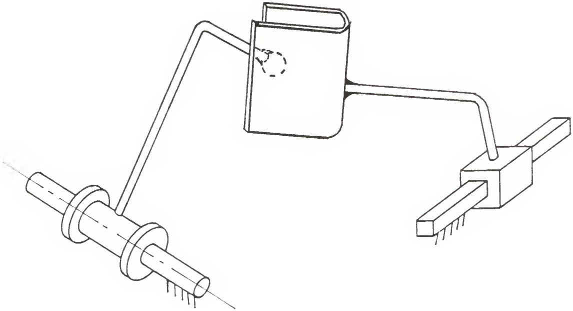


Figure 3.2 The revolute—sphere-plane—prismatic (R-Sp-R) mechanism

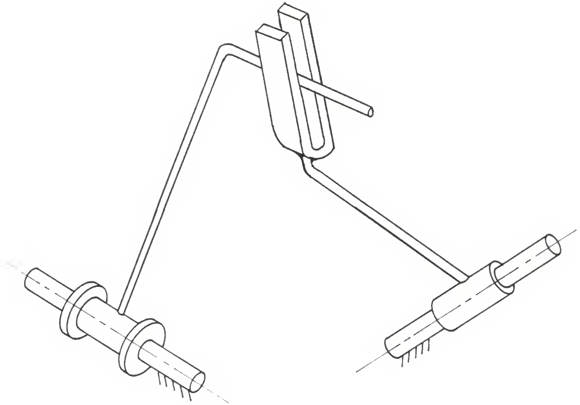


Figure 3.3 The revolute--cylinder-plane-cylinder (R-Cp-C) mechanism

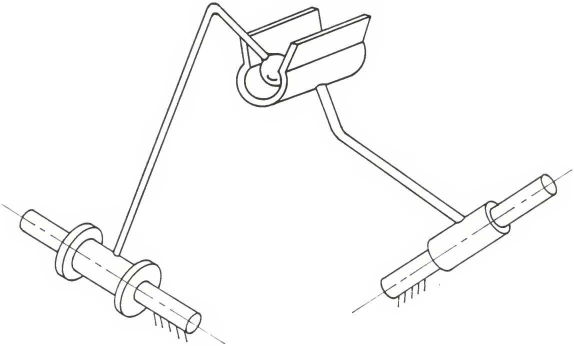


Figure 3.4 The revolute--sphere-groove--cylinder (R-Sg-C) mechanism

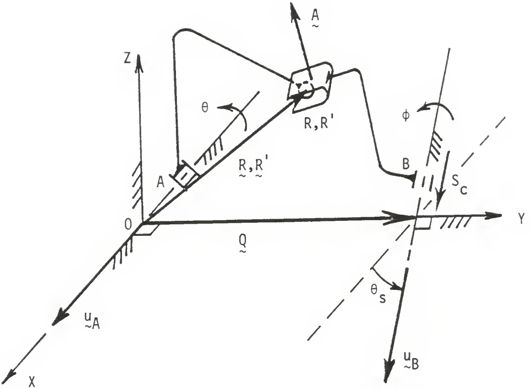


Figure 3.5 The R-Sp-C mechanism and the associated vectors and scalars

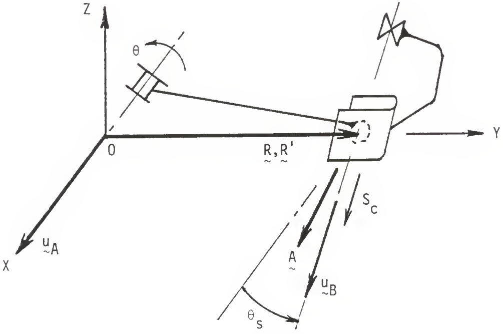


Figure 3.5a The R-Sp-P mechanism and the associated vectors and scalars

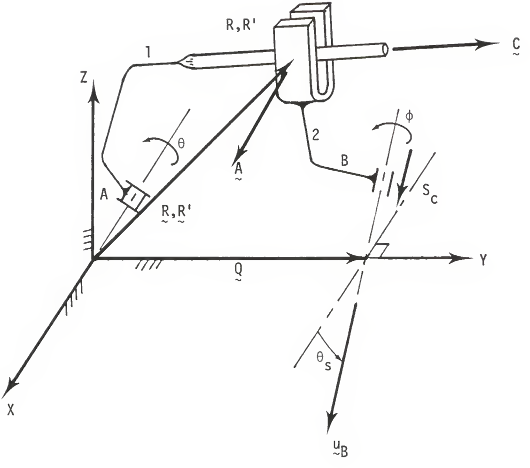


Figure 3.6 The R-Cp-C mechanism and the associated vectors and scalars

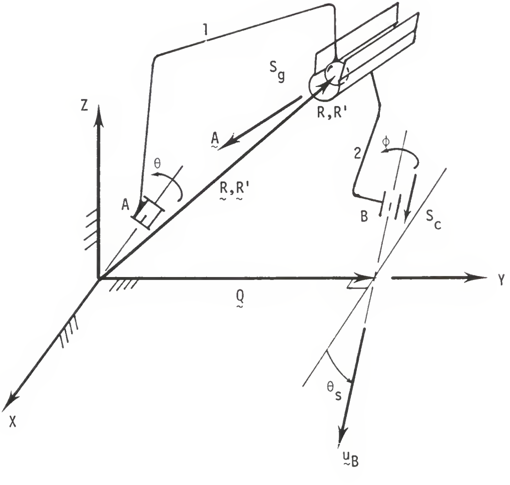


Figure 3.7 The R-Sg-C mechanism with the associated vectors and scalars

$\underset{\sim}{R} = \underset{\sim}{R}'$ is also true at the initial position. The figure also shows the unit vector $\underset{\sim}{C}$ which defines the axis of the cylinder in the cylinder-plane pair.

For the R-Sg-C mechanism in figure 3.7, vector $\underset{\sim}{A}$ is in the direction of the groove of the sphere-groove pair, and the scalar quantity S_g is the translation of the sphere along the groove. The descriptions of the other vectors and scalars are the same as in the R-Sp-C mechanism of figure 3.5.

3.3 Parameters Available for Synthesis

The vectors shown in figures 3.5 to 3.8 can be classified as position vectors or as design vectors. The vectors $\underset{\sim}{u}_A$ and $\underset{\sim}{Q}$ are the position vectors since they simply locate and orient the three-link mechanisms within a spatial coordinate system. On the other hand, the vectors $\underset{\sim}{R}$, $\underset{\sim}{A}$, $\underset{\sim}{u}_B$ and $\underset{\sim}{C}$ are the design vectors since they define the geometry of the mechanism (recall that $\underset{\sim}{R} = \underset{\sim}{R}'$). Thus, any set of these design vectors will define a unique mechanism.

The vector $\underset{\sim}{Q}$ and $\underset{\sim}{R}$ of the R-Sp-P mechanism are specified without loss of generality to be of unit lengths. This fixes the scale of the mechanism to be synthesized. Subject to other design considerations, the synthesized mechanism can then be scaled up or down and still preserve the required function generation properties.

In most cases, whenever a function generation problem is specified, the skew angle θ_s between two axes of rotation or direction of translation is usually specified. On a more important note, it will be evident from the next chapter that making θ_s a design parameter greatly increases the degree of the synthesis equations. Although solutions can be obtained for these high degree equations by using numerical methods, this would deviate from the objective of obtaining closed-form solutions. This brings to light the paramount advantage of using closed-form solutions--it presents all the possible solutions including complex roots (though they are not useful). Therefore, θ_s will always be considered given in the function generation synthesis of the three-link spatial mechanisms.

For the R-Sp-R mechanism, the design vectors for synthesis are $\underset{\sim}{R}$ and $\underset{\sim}{A}$ (see figure 3.5). Vector $\underset{\sim}{R}$ locates a point in space and is specified by three coordinates while $\underset{\sim}{A}$ is a unit vector and is thus defined by two independent parameters. This gives five parameters available for synthesis.

For the R-Sp-P mechanism, only the two parameters of vector $\underset{\sim}{A}$ are available for synthesis, since the vector $\underset{\sim}{R}$ is considered a positioning and scaling vector.

For the R-Cp-P mechanism, the design vectors are $\underset{\sim}{R}$, $\underset{\sim}{A}$ and $\underset{\sim}{C}$. Unlike the vector $\underset{\sim}{R}$ in the R-Sp-R, R-Sp-P and R-Sg-C mechanisms, which locate a point (sphere center) in space, the vector $\underset{\sim}{R}$ in the R-Cp-C mechanism locates a vector in space. A point in space is defined by three coordinates while a free vector can be located in space by only two independent coordinates. Figure 3.8 shows the vector $\underset{\sim}{C}$ intersecting the X-Z plane and located by vectors $\underset{\sim}{R}_o$ and $\underset{\sim}{R}_a$. Note that $\underset{\sim}{R}_o$ is defined by only two coordinates (r_x and r_z) and the vector $\underset{\sim}{R}_a$, although it has three coordinates, is determined from the sum of $\underset{\sim}{R}_o$ and $\underset{\sim}{C}$ times a scalar.

Instead of locating the vector $\underset{\sim}{C}$ by its intersection with a plane, the vector $\underset{\sim}{R}$ will still be used but with one of its coordinates specified arbitrarily. The reason for this is because vector $\underset{\sim}{C}$ is to be solved for and thus, its intersection with any plane cannot be known beforehand. Thus, the R-Cp-C mechanism is defined by six design parameters--two of the three coordinates of $\underset{\sim}{R}$ and two unit vector parameters each for $\underset{\sim}{A}$ and $\underset{\sim}{C}$.

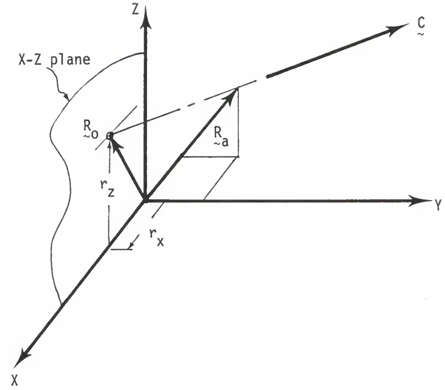


Figure 3.8 Location of a vector in space

For the R-Sg-C mechanism, the design vectors $\underset{\sim}{R}$ and $\underset{\sim}{A}$ presents five parameters available for design.

The three-link mechanisms with their design vectors and number of parameters available for design are listed as follows:

Mechanism	Design Vectors	Parameters Available for Synthesis
R-Sp-R	$\underset{\sim}{R}$, $\underset{\sim}{A}$	5
R-Sp-P	$\underset{\sim}{A}$	2
R-Cp-C	$\underset{\sim}{R}$, $\underset{\sim}{A}$, $\underset{\sim}{C}$	6
R-Sg-C	$\underset{\sim}{R}$, $\underset{\sim}{A}$	5

3.4 Pair Geometry Constraint Equations

The basis of the equations of motion of the different mechanisms (and in turn, the synthesis and analysis equations) is the pair geometry constraint equations. These equations give the motion of the pair elements as restricted by the geometry of the pair.

The vectorial representation will be used to define these constraint equations because using vectors provides the advantage of being visualized even in three dimensions. Also, during the derivations, vector manipulations can be used to great advantage in simplifying equations.

3.4.1 The Sphere-Plane Pair Constraint Equation

Figure 3.9 shows the Sp pair elements at an initial position defined by vectors $\underset{\sim}{R}$, $\underset{\sim}{R}'$ and $\underset{\sim}{A}$, and at a displaced (jth) position, now defined by vectors $\underset{\sim}{R}_j$, $\underset{\sim}{R}'_j$ and $\underset{\sim}{A}_j$. The sphere-plane pair constraint equation is obtained from the fact that any relative displacement between the two points, R which is fixed to the sphere and R' which is fixed to the plane, will always be in the plane and perpendicular to the vector $\underset{\sim}{A}$.

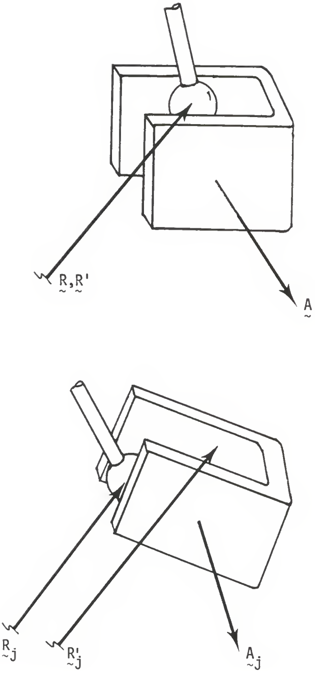


Figure 3.9 Initial and (jth) positions of the Sp pair

Since the pair elements have moved to their respective (jth) positions, this is expressed in the scalar dot products of:

$$(\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \underset{\sim}{A}_j = 0 \qquad (3.4)$$

3.4.2 The Cylinder-Plane Pair Constraint Equation

Figure 3.10 shows the Cp pair in the initial and (jth) position. The motion between the cylinder and the plane is characterized by two translations of the cylinder with respect to the plane plus two rotations of the cylinder about axis $\underset{\sim}{A}$ and axis $\underset{\sim}{C}$. Thus, the relative displacement between points R and R' must be perpendicular to the vector $\underset{\sim}{A}$ at any displaced position and that the vectors $\underset{\sim}{A}$ and $\underset{\sim}{C}$ must always be perpendicular to each other. Expressed as two separate equations, the pair constraint equations are:

$$(\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \underset{\sim}{A}_j = 0 \qquad (3.5)$$

and

$$\underset{\sim}{A}_j \cdot \underset{\sim}{C}_j = 0 \qquad (3.6)$$

3.4.3 The Sphere-Groove Pair Constraint Equation

Figure 3.11 shows the Sg pair at an initial position and at a (jth) position. The motion of the sphere relative to the groove is made up of a linear translation in the groove and three rotations of the sphere. The relative displacement of points R and R' in this pair must be in the direction of the groove. Written in an equation form, this constrained motion is given by:

$$\underset{\sim}{R}_j - \underset{\sim}{R}'_j = S_{gj}\underset{\sim}{A}_j \qquad (3.7)$$

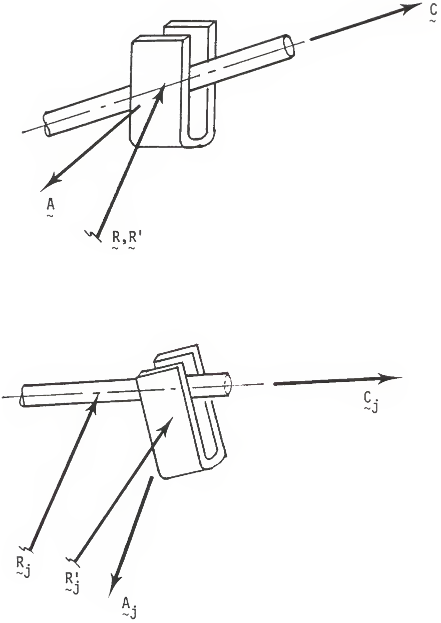


Figure 3.10 Initial and (jth) positions of the Cp pair

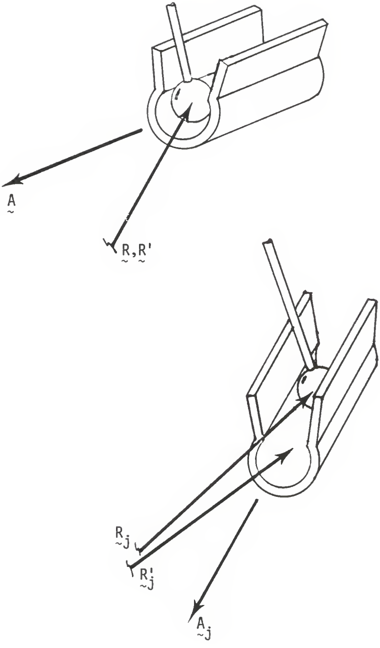


Figure 3.11 Initial and (jth) positions of the Sg pair

where: S_{gj} is a scalar that defines the displacement of the sphere in the direction of $\tilde{A}_j$ after a (jth) displacement.

3.5 Derivation of Vector Displacements

Section 3.4 presented the pair geometry constraint equations. The task now is to obtain the equation of motion of the mechanisms containing these higher pairs considered - the Sp pair, the Cp pair and the Sg pair. As expressed in the pair geometry constraint equation, the expressions for the (jth) position of the points R and R' and also the orientation of the vector $\tilde{A}$ after a (jth) displacement must all be known.

The procedure for obtaining the equation of motion for the mechanisms is as follows:

1. The two moving links of the mechanism are separated at the higher pair.
2. Using vector rotations, the new orientation of the vectors $\tilde{A}$ and $\tilde{C}$ (in the case of the R-Cp-C) defined by $\tilde{A}_j$ and $\tilde{C}_j$ are obtained after the respective links have moved to a (jth) position. The expressions for vectors $\tilde{R}_j$ and $\tilde{R}'_j$ which define the locations of points R and R' are to be obtained.
3. The two moving links are then combined by imposing the pair geometry constraint equations on the expressions that have been obtained separately.

Figure 3.12 shows the R-Sp-C mechanism separated at the sphere-plane pair showing the new positions of points R, R' and the new orientation and direction of the vector $\tilde{A}$. The foregoing derivations

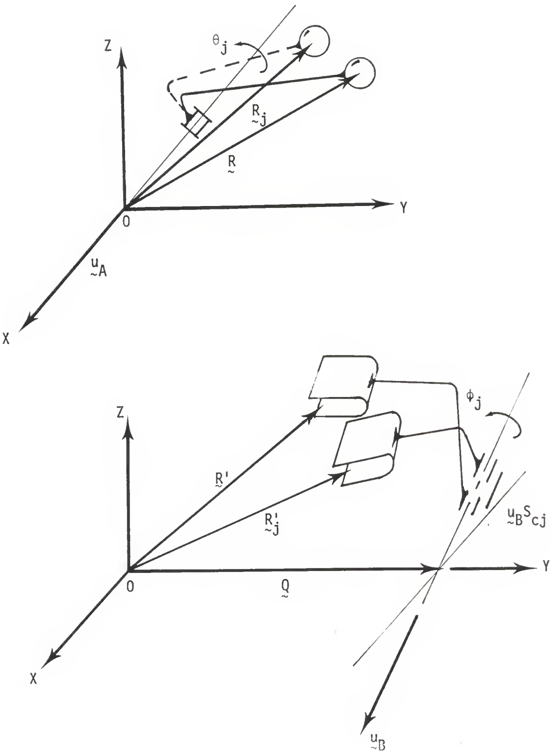


Figure 3.12 The R-Sp-C mechanism separated at the Sp pair

are not limited to the R-Sp-C mechanism or the other two mechanisms (R-Cp-C and R-Sg-C) but are general for any vector displacements as well.

3.5.1 Vector Rotations

Figure 3.13 shows the vector $\underset{\sim}{A}_j$ which is the new orientation and direction of the vector $\underset{\sim}{A}$ after it was rotated by an angle of ϕ_j about an axis defined by $\underset{\sim}{u}_B$. Since $\underset{\sim}{A}$ is a free vector, it can be moved while preserving orientation and direction until it intersects the axis defined by $\underset{\sim}{u}_B$. From figure 3.14, $\underset{\sim}{A}_j$ is obtained as

$$\underset{\sim}{A}_j = (\underset{\sim}{A} \cdot \underset{\sim}{u}_B)\underset{\sim}{u}_B + \cos\phi_j[\underset{\sim}{A} - (\underset{\sim}{A} \cdot \underset{\sim}{u}_B)\underset{\sim}{u}_B] + \sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A}) \qquad (3.8)$$

Adding and subtracting $\underset{\sim}{A}$ from equation (3.8), using the following identity for any two vectors $\underset{\sim}{C}$ and $\underset{\sim}{D}$:

$$(\underset{\sim}{C} \times \underset{\sim}{D}) \times \underset{\sim}{C} = \underset{\sim}{D} - (\underset{\sim}{D} \cdot \underset{\sim}{C})\underset{\sim}{C} \qquad (3.9)$$

and introducing the following notation for any two vectors $\underset{\sim}{u}_C$ and $\underset{\sim}{D}$:

$$(\underset{\sim}{u}_C \times \underset{\sim}{D}) \times \underset{\sim}{u}_C = \underset{\sim}{U}_{CD} \qquad (3.10)$$

equation (3.8) is simplified as:

$$\underset{\sim}{A}_j = \underset{\sim}{A} + (\cos\phi_j - 1)\underset{\sim}{U}_{BA} + \sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A}) \qquad (3.11)$$

3.5.2 Screw Displacement of a Point in Vector Form

Figure 3.15 shows the new location of the point R' at link 2 as defined by $\underset{\sim}{R}'_j$ after it has rotated about the axis defined by $\underset{\sim}{u}_B$ and translated along the same axis by the angle ϕ_j and S_{Cj} respectively from the initial location defined by R'. This combination of linear and rotational motion is the resulting general motion of link 2 grounded by a cylindric (C) pair. Note that the rotation and translation

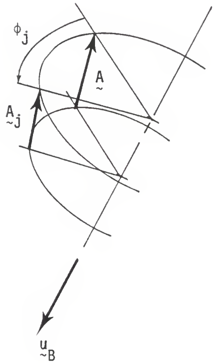


Figure 3.13 The (jth) position of vector $\underset{\sim}{A}$

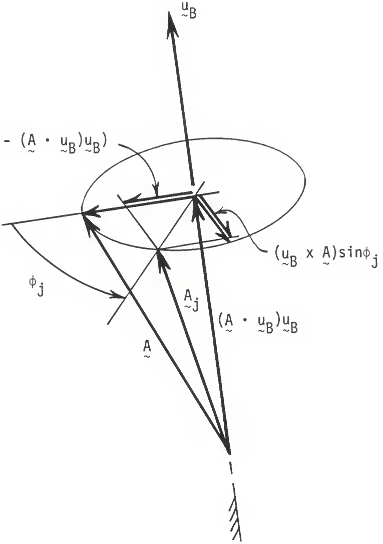

$\underset{\sim}{u}_B$
$\cos\phi_j(\underset{\sim}{A} - (\underset{\sim}{A} \cdot \underset{\sim}{u}_B)\underset{\sim}{u}_B)$
$(\underset{\sim}{u}_B \times \underset{\sim}{A})\sin\phi_j$
ϕ_j
$\underset{\sim}{A}_j$
$(\underset{\sim}{A} \cdot \underset{\sim}{u}_B)\underset{\sim}{u}_B$
$\underset{\sim}{A}$

Figure 3.14 Rotation of $\underset{\sim}{A}$ about $\underset{\sim}{u}_B$ by ϕ_j

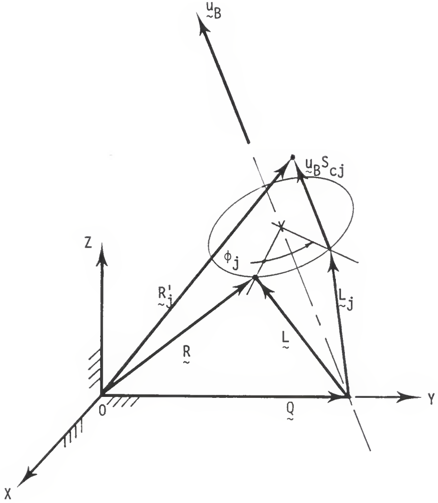


Figure 3.15 (Jth) position of point R'

in this form of screw displacement is commutative. From the same figure 3.15, $\underset{\sim}{R}'_j$ can be written as:

$$\underset{\sim}{R}'_j = \underset{\sim}{Q} + \underset{\sim}{L}_j + S_{cj}\underset{\sim}{u}_B \qquad (3.12)$$

where: $\underset{\sim}{L}_j$ is the vector $\underset{\sim}{R} - \underset{\sim}{Q}$ rotated about $\underset{\sim}{u}_B$ by an angle ϕ_j. Having previously obtained an expression for a vector rotation in equation (3.11), the expression for $\underset{\sim}{L}_j$ is obtained as:

$$\underset{\sim}{L}_j = \underset{\sim}{L} + (\cos\phi_j - 1)\underset{\sim}{U}_{BL} + \sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L}) \qquad (3.13)$$

Substituting equation (3.13) and $(\underset{\sim}{L} - \underset{\sim}{R})$ for $\underset{\sim}{Q}$ into equation (3.12) results in the expression for the displaced position of point R':

$$\underset{\sim}{R}'_j = \underset{\sim}{R} + (\cos\phi_j - 1)\underset{\sim}{U}_{BL} + \sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L}) + S_{cj}\underset{\sim}{u}_B \qquad (3.14)$$

The expression for $\underset{\sim}{R}_j$ which gives the new location of the point R fixed in link 1 after only a rotation of θ_j (since the grounded pair is a revolute (R) pair with axis $\underset{\sim}{u}_A$) is obtained in a similar manner. From figure 3.16, $\underset{\sim}{R}_j$ is simply the vector $\underset{\sim}{R}$ rotated about $\underset{\sim}{u}_A$ by an angle θ_j. Thus:

$$\underset{\sim}{R}_j = \underset{\sim}{R} + (\cos\theta_j - 1)\underset{\sim}{U}_{AR} + \sin\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) \qquad (3.15)$$

3.6 R-Sp-R and R-Sp-P Equations of Motion

For the R-Sp-R mechanism, equations (3.11), (3.14) and (3.15) are substituted in the sphere-plane pair constraint given by equation (3.4). This time, however, the term $S_{cj}\underset{\sim}{u}_B$ is made zero in the expression for $\underset{\sim}{R}'_j$ to make the grounded pair of link 2 into a revolute (R) pair. The resultant equation of motion for the R-Sp-R mechanism is:

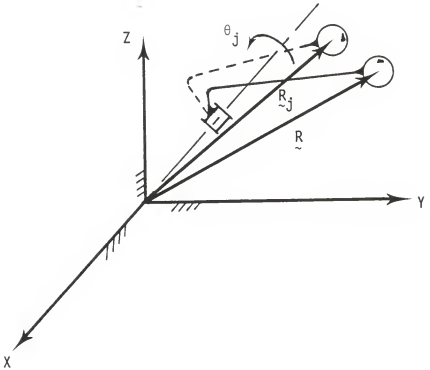


Figure 3.16 (Jth) position of the sphere center R

$$[(\cos\theta_j - 1)\underset{\sim}{U}_{AR} + \sin\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) - (\cos\phi_j - 1)\underset{\sim}{U}_{BL} - \sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L})] \cdot$$

$$[\underset{\sim}{A} + (\cos\phi_j - 1)\underset{\sim}{U}_{BA} + \sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A})] = 0 \qquad (3.16)$$

The R-Sp-P equation of motion is obtained by modifying equations (3.11) and (3.14) by setting ϕ_j to zero and substituting them together with equation (3.15) into the sphere-plane pair constraint equation. This gives us:

$$[(\cos\theta_j - 1)\underset{\sim}{U}_{AR} + \sin\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) - S_{cj}\underset{\sim}{u}_B] \cdot \underset{\sim}{A} = 0 \qquad (3.17)$$

3.7 The R-Cp-C Equation of Motion

The equation of motion for the R-Cp-C mechanism requires the expressions for $\underset{\sim}{R}_j$, $\underset{\sim}{R}'_j$, $\underset{\sim}{A}_j$ and also of $\underset{\sim}{C}_j$. The expression for $\underset{\sim}{C}_j$ is derived in a manner similar to $\underset{\sim}{A}_j$ but with the appropriate terms. This gives:

$$\underset{\sim}{C}_j = \underset{\sim}{C} + (\cos\theta_j - 1)\underset{\sim}{U}_{AC} + \sin\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{C}) \qquad (3.18)$$

Substituting these expressions for $\underset{\sim}{R}_j$, $\underset{\sim}{R}'_j$, $\underset{\sim}{A}_j$ and $\underset{\sim}{C}_j$ into the cylinder-plane pair constraint equations (3.5) and (3.6), results in the R-Cp-C equation of motion:

$$[\underset{\sim}{A} + (\cos\phi_j - 1)\underset{\sim}{U}_{BA} + \sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A})] \cdot [\underset{\sim}{C} + (\cos\theta_j - 1)\underset{\sim}{U}_{AC} +$$

$$\sin\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{C})] = 0 \qquad (3.19)$$

and

$$[(\cos\theta_j - 1)\underset{\sim}{U}_{AR} + \sin\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) - (\cos\phi_j - 1)\underset{\sim}{U}_{BL} - \sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L})$$

$$- S_{cj}\underset{\sim}{u}_B] \cdot [\underset{\sim}{A} + (\cos\phi_j - 1)\underset{\sim}{U}_{BA} + \sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A})] = 0 \qquad (3.20)$$

3.8 The R-Sg-C Equation of Motion

Substituting equations (3.11), (3.14) and (3.15) into the sphere-groove pair constraint equation (3.7) gives the R-Sg-C equation of motion as follows:

$$[(\cos\theta_j - 1)\underset{\sim}{U}_{AR} + \sin\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) - (\cos\phi_j - 1)\underset{\sim}{U}_{BL} - \sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L}) - S_{cj}\underset{\sim}{u}_B]$$

$$= S_{gj}[\underset{\sim}{A} + (\cos\phi_j - 1)\underset{\sim}{U}_{BA} + \sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A})] \qquad (3.21)$$

The equations of motion obtained so far for the three-link mechanisms considered are the displacement equations. In order to obtain the higher order equations of motion, the time derivatives of these displacement equations must be obtained. It will be shown in the next chapter that it is more convenient to carry out these differentiations with respect to time after the displacement synthesis equations have been derived.

3.9 Number of Positions Available for Synthesis

The three-link mechanisms considered each have different numbers of parameters available for synthesis and involve different forms of the constraint equation. The interrelation of this number of parameters and the constraint equations determines the maximum number of precision points available for synthesis for each of the three-link mechanisms considered.

For the R-Sp-R or R-Sp-P mechanism, there are respectively five and two parameters available for design and one scalar constraint equation. From this, one equation can be written for each displaced position from the initial position. To obtain a determinate solution, a maximum of five and two equations respectively, can then be written to obtain the required five or two parameters that will define the geometry of the mechanisms at the starting position. Therefore, the R-Sp-R mechanism can be synthesized for the maximum number of six precision points while the R-Sp-P mechanism can be synthesized for three precision points. Expressed in an equation, the maximum number of precision points for the synthesis of the R-Sp-R mechanism is six as obtained from

$$e(n - 1) = 5 \tag{3.22}$$

$$n = 6$$

where $e = 1$ i.e., one equation for each displaced position

For the R-Cp-C mechanism, there are six design parameters available for synthesis. Equations (3.5) and (3.6) are to be written together at any (jth) displaced position from the initial position. In addition to this, equation (3.6) must also be satisfied at the initial position, i.e.

$$\underset{\sim}{A} \cdot \underset{\sim}{C} = 0 \tag{3.23}$$

Thus, for every displaced position, two scalar equations are written plus the single initial position equation (3.23). The maximum number of precision points is obtained from the following equation:

$$e(n - 1) + 1 = 6 \tag{3.24}$$

where $e = 2$ for the number of equations written for each displaced position.

This leads to $n = 3\text{-}1/2$ which indicates that the R-Cp-C mechanism can be synthesized for a maximum number of three-precision points.

In the case of the R-Sg-C mechanism, there are five parameters available for synthesis. The constraint equation (3.7) is a vector equation and is made up of three scalar equations that are to be written for each displaced position. However, the set of three scalar equations has the addition unknown S_{gj} for each displaced position. The maximum number of precision points for synthesis is solved in the following equation:

$$3(n - 1) = 5 + (n - 1) \qquad (3.25)$$

This equation gives n = 3-1/2 which indicates that the R-Sg-C mechanism can be synthesized for a maximum number of three positions only.

CHAPTER 4

EQUATIONS FOR MSP SYNTHESIS

In this chapter, the synthesis equations, explicit in the design parameters, will be derived for the four three-link function generators namely: R-Sp-R, Rp-Sp-P, R-Cp-C and the R-Sg-C mechanisms. The derivations are based on the equations of motions and the required expressions for the different vectors. The resulting general equations are later simplified by considering the appropriate parameters for the synthesis of each particular mechanism.

4.1 Vector Expressions

The position vectors $\underset{\sim}{u}_A$, $\underset{\sim}{u}_B$ and $\underset{\sim}{Q}$ are written with their coordinates specified as

$$\underset{\sim}{u}_A = 1\hat{i} + 0\hat{j} + 0\hat{k} \tag{4.1}$$

$$\underset{\sim}{u}_B = b_1\hat{i} + 0\hat{j} + b_3\hat{k} \tag{4.2}$$

$$\underset{\sim}{Q} = 0\hat{i} + 1\hat{j} + 0\hat{k} \tag{4.3}$$

where

$$\left.\begin{aligned} b_1 &= \cos\theta_s \\ b_3 &= -\sin\theta_s \end{aligned}\right. \tag{4.4}$$

On the other hand, the design vectors are expressed in their coordinates as

$$\underset{\sim}{R} = r_1\hat{i} + r_2\hat{j} + r_3\hat{k} \tag{4.5}$$

$$\underset{\sim}{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \tag{4.6}$$

and in the R-Cp-C case,

$$\underset{\sim}{C} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \tag{4.7}$$

where

$$a_1^2 + a_2^2 + a_3^2 = 1$$

$$c_1^2 + c_2^2 + c_3^2 = 1 \tag{4.8}$$

Using equations (4.1) to (4.7), the expressions for the terms in the equations of motions are derived as

$$\underset{\sim}{u}_A \times \underset{\sim}{R} = 0\hat{i} - r_3\hat{j} + r_2\hat{k} \tag{4.9}$$

$$\underset{\sim}{U}_{AR} = 0\hat{i} + r_2\hat{j} + r_3\hat{k} \tag{4.10}$$

$$\underset{\sim}{u}_B \times \underset{\sim}{L} = b_3(1 - r_2)\hat{i} + (b_3r_1 - b_1r_3)\hat{j} + b_1(r_2 - 1)\hat{k} \tag{4.11}$$

$$\underset{\sim}{U}_{BL} = (b_3^2r_1 - b_1b_3r_3)\hat{i} + (r_2 - 1)\hat{j} + (b_1^2r_3 - b_1b_3r_1)\hat{k} \tag{4.12}$$

$$\underset{\sim}{u}_B \times \underset{\sim}{A} = -b_3a_2\hat{i} + (b_3a_1 - b_1a_3)\hat{j} + b_1a_2\hat{k} \tag{4.13}$$

$$\underset{\sim}{U}_{BA} = (b_3^2a_1 - b_1b_3a_3)\hat{i} + a_2\hat{j} + (b_1^2a_3 - b_1b_3a_1)\hat{k} \tag{4.14}$$

$$\underset{\sim}{u}_A \times \underset{\sim}{C} = 0\hat{i} - c_3\hat{j} + c_2\hat{k} \tag{4.15}$$

$$\underset{\sim}{U}_{AC} = 0\hat{i} + c_2\hat{j} + c_3\hat{k} \tag{4.16}$$

Equations (4.9) to (4.16) are then substituted into the respective equations of motion of the mechanisms to obtain the displacement synthesis equations. The equations for MSP synthesis are then obtained by generalizing the synthesis equations to include time derivative expressions.

4.2 The MSP Synthesis Equation for the R-Sp-R Mechanism

The equation of motion of the R-Sp-R mechanism is recalled as

$$[C_1\theta_j\underset{\sim}{U}_{AR} + S\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) - C_1\phi_j\underset{\sim}{U}_{BL} - S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L})] \cdot$$

$$[\underset{\sim}{A} + C_1\phi_j\underset{\sim}{U}_{BA} + S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A})] = 0 \qquad (4.17)$$

Substituting the vector expressions derived in section 4.1 into equation of motion (4.17) and rearranging the terms will result in the following equation:

$$\{[-C_1\phi_j b_3{}^2 r_1 + S\phi_j b_3 r_2 + C_1\phi_j b_1 b_3 r_3 - S\phi_j b_3]\hat{i} +$$

$$[-S\phi_j b_3 r_1 + (C_1\theta_j - C_1\phi_j)r_2 + (S\phi_j b_1 - S\theta_j)r_3 + C_1\phi_j]\hat{j} +$$

$$[C_1\phi_j b_1 b_3 r_1 + (S\theta_j - S\phi_j b_1)r_2 + (C_1\theta_j - C_1\phi_j b_1{}^2)r_3 + S\phi_j b_1]\hat{k}\} \cdot$$

$$\{[(1 + C_1\phi_j b_3{}^2)a_1 - S\phi_j b_3 a_2 - C_1\phi_j b_1 b_3 a_3]\hat{i} +$$

$$[S\phi_j b_3 a_1 + (1 + C_1\phi_j)a_2 - S\phi_j b_1 a_3]\hat{j} +$$

$$[-C_1\phi_j b_1 b_3 a_1 + S\phi_j b_1 a_2 + (1 + C_1\phi_j b_1{}^2)a_3]\hat{k}\} = 0 \qquad (4.18)$$

where

$$C_1\theta_j = \cos\theta_j - 1$$

$$C_1\phi_j = \cos\phi_j - 1$$

$$S\theta_j = \sin\theta_j \qquad (4.19)$$

$$S\phi_j = \sin\phi_j$$

Carrying out the dot product operation and combining terms will result in the following form of the displacement equation.

$$K_{1j}r_1a_1 + K_{2j}r_1a_2 + pK_{1j}r_1a_3 + K_{3j}r_2a_1 + K_{4j}r_2a_2 + K_{5j}r_2a_3 + K_{6j}r_3a_1$$

$$+ K_{7j}r_3a_2 + K_{8j}r_3a_3 + K_{2j}a_1 + qK_{1j}a_2 + pK_{2j}a_3 = 0 \tag{4.20}$$

$$\text{where} \qquad p = -b_1/b_3 \quad \text{and} \quad q = -1/b_3^{\,2} \tag{4.21}$$

Dividing equation (4.20) by a_1 will normalize the coordinates of the vector $\underset{\sim}{A}$ so that the displacement equation is now

$$[(1 + pd_2)k_{1j} + K_{2j}d_1]r_1 + (K_{3j} + K_{4j}d_1 + K_{5j}d_2)r_2 +$$

$$(K_{6j} + K_{7j}d_1 + K_{8j}d_2)r_3 + [(1 + pd_2)K_{2j} + qK_{1j}d_1] = 0 \tag{4.22}$$

where $d_1 = a_2/a_1$ and $d_2 = a_3/a_1$ are chosen as the parameters of $\underset{\sim}{A}$ and the K_{ij}-s are listed in Appendix B

Since the K_{ij} terms are functions only of θ_j, ϕ_j and θ_s, equation (4.22) can be used as a displacement or FSP synthesis equation by specifying θ_j's and ϕ_j's and solving for the design parameters r_i-s and d_k-s.

Equation (4.22) is the displacement equation for the R-Sp-R mechanism in a different form. Its time derivatives can be obtained to give the velocity, acceleration and even higher order relations. Taking the time derivatives of equation (4.22) will not change its form with respect to the variables r_1, r_2, r_3, d_1 and d_2 since these r_i-s and d_k-s are constants that define the initial position of the mechanism. In short, the n-th time derivative of equation (4.22) is the n-th time derivatives of the K_{ij} terms multiplied by the same corresponding r_i-s and d_k-s. Using the superscript n to denote the n-th time derivative, the multiply separated position (MSP) synthesis equation is

$$[(1 + pd_2)K_{1j}^{n} + K_{2j}^{n}d_1]r_1 + (K_{3j}^{n} + K_{4j}^{n}d_1 + K_{5j}^{n}d_2)r_2 + (K_{6j}^{n} + K_{7j}^{n}d_1 + K_{8j}^{n}d_2)r_3 + [(1 + pd_2)K_{2j}^{n} + qK_{1j}^{n}d_1] = 0 \qquad (4.23)$$

$$j = 2, 3, 4 \text{ up to } 6$$

$$n = 0 \text{ or } 1 \text{ or } 2 \text{ or } 3\ldots$$

An important feature of equation (4.23) is that finitely separated positions (FSP) or infinitesimally separated positions (ISP) or combinations of the two (MSP) can be specified with ease by using the appropriate K_{ij}^{n} terms in the equation. The counter j in the equation takes on a slightly different meaning -- it is now a label for either ISP's or FSP's. The initial position is also set as the j = 1 position. For every (jth) position (except j = 1), equation (4.23) is written by computing each K_{ij}^{n} term from the required functional relations of $[\theta_j, \dot{\theta}_j, \ddot{\theta}_j \ldots, \phi_j, \dot{\phi}_j, \ddot{\phi}_j \ldots]$. Whenever ISP's are specified, other lower order function generation parameters must also be specified at that discrete position. For example, if an ISP of second order is specified, then the displacements and velocities together with the accelerations must be specified. This is further evident in the expressions for the K_{ij}^{n} terms derived and listed in the appendix. If P, V and A stand for position, velocity and acceleration, respectively, and a dash represents separations between discrete positions, then, only clusters of P, PV or PVA can be grouped between dashes. Groups like PA or VA or V alone are not valid. An example of a valid set of specifications is the representation of the following six MSP specifications: PV-P-PVA. Note also, that the j= 1 position refers to the initial position where θ_1 and ϕ_1 are both zero.

Finally, the system of equations obtained from equation (4.23) (written for each of the different (jth) positions) can be solved for the unknown design parameters r_i-s and/or d_k-s.

4.3 The MSP Synthesis Equation for the R-Sp-P Mechanism

The equation of motion for the R-Sp-P mechanism from Chapter 3 is

$$[C_1\theta_j \underset{\sim}{U}_{AR} + S\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) - S_{cj}\underset{\sim}{u}_B] \cdot \underset{\sim}{A} = 0 \qquad (4.24)$$

Substituting the vector expressions derived for the terms in equation (4.24), setting r_1, r_3 equal to zero, and r_2 to one, carrying out the dot product, normalizing the coordinates of the vector $\underset{\sim}{A}$ and taking the time derivatives will result in the MSP synthesis equation

$$P_{1j}^{n} d_1 + P_{2j}^{n} d_2 = P_{3j}^{n} \qquad (4.25)$$

$$j = 2,\ 3$$

$$n = 0 \text{ or } 1 \text{ or } 2$$

The P_{ij}^{n} terms are deterministic functions of the motion parameters and are listed below for n = 0, n = 1 and n = 2

$$\begin{aligned} P_{1j} &= \cos\theta_j - 1 \\ P_{1j}^{1} &= -\sin\theta_j\dot{\theta}_j \\ P_{1j}^{2} &= -(\cos\theta_j\dot{\theta}_j^{2} + \sin\theta_j\ddot{\theta}_j) \end{aligned} \qquad (4.25a)$$

$$\begin{aligned} P_{3j} &= b_1 S_{cj} \\ P_{3j}^{1} &= b_1 \dot{S}_{cj} \\ P_{3j}^{2} &= b_1 \ddot{S}_{cj} \end{aligned} \qquad (4.25c)$$

$$\begin{aligned} P_{2j} &= \sin\theta - b_3 S_{cj} \\ P_{2j}^{1} &= \cos\theta_j\dot{\theta}_j - b_3\dot{S}_{cj} \\ P_{2j}^{2} &= -(\sin\theta_j\dot{\theta}_j^{2} - \cos\theta_j\ddot{\theta}_j) - b_3\ddot{S}_{cj} \end{aligned} \qquad (4.25b)$$

4.4 The MSP Synthesis Equation for the R-Cp-C Mechanism

Recalling from Chapter 3, the displacement equations for the R-Cp-C mechanism are

$$[\underset{\sim}{A} + C_1\phi_j\underset{\sim}{U}_{BA} + S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A})] \cdot [\underset{\sim}{C} + C_1\theta_j\underset{\sim}{U}_{AC} + S\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{C})] = 0 \tag{4.26}$$

and

$$[C_1\theta_j\underset{\sim}{U}_{AR} + S\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) - C_1\phi_j\underset{\sim}{U}_{BL} - S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L}) - \underset{\sim}{u}_B S_{cj}] \cdot$$

$$[\underset{\sim}{A} + C_1\phi_j\underset{\sim}{U}_{BA} + S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A})] = 0 \tag{4.27}$$

In addition to these two equations, the following initial position condition must also be satisfied:

$$\underset{\sim}{A} \cdot \underset{\sim}{C} = 0 \tag{4.28}$$

The same procedure to obtain the synthesis equation for the R-Sp-R mechanism is also used here. This results in the following MSP synthesis equations for the R-Cp-C mechanism.

$$e_1d_1 + e_2d_2 + 1 = 0 \tag{4.29}$$

$$L_{1j}^n + L_{2j}^n e_1 + L_{3j}^n e_2 + L_{4j}^n d_1 + L_{5j}^n d_1 e_1 + L_{6j}^n d_1 e_2 + L_{7j}^n d_2$$

$$+ L_{8j}^n d_2 e_1 + L_{9j}^n d_2 e_2 = 0 \tag{4.30}$$

$$[(1 + pd_2)M_{1j}^n + M_{2j}^n d_1]r_1 + [M_{3j}^n + M_{4j}^n d_1 + M_{5j}^n d_2]r_2$$

$$+ [M_{6j}^n + M_{7j}^n d_1 + M_{8j}^n d_2]r_3 + [M_{9j}^n + qM_{1j}^n d_1 + M_{10j}^n d_2] = 0 \tag{4.31}$$

where

$$\begin{aligned} d_1 &= a_2/a_1 \\ d_2 &= a_3/a_1 \\ e_1 &= c_2/c_1 \\ e_2 &= c_3/c_1 \end{aligned} \qquad (4.32)$$

The L_{ij}^n and M_{ij}^n terms are listed in the appendix for n = 0, n = 1 and n = 2.

4.5 The MSP Synthesis Equation For The R-Sg-C Mechanism

The synthesis equations for the R-Sg-C mechanism are derived by substituting the vector expressions (4.1) to (4.16) in the equation of motion

$$C_1\theta_j\underset{\sim}{U}_{AR} + S\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) - C_1\phi_j\underset{\sim}{U}_{BL} - S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L}) - S_{cj}\underset{\sim}{u}_B =$$

$$S_{gj}[\underset{\sim}{A} + C_1\phi_j\underset{\sim}{U}_{BA} + S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A})] \qquad (4.33)$$

The resulting displacement synthesis equation in matrix form is

$$[G_{ikj}]\ \{r_1\} + S_{gj}[H_{ikj}]\ \{a_i\} = \{P_{ij}\} \qquad (4.34)$$

$$i = 1, 2, 3$$

$$j = 2, 3$$

The elements of the (3 x 3) matrices $[G_{ikj}]$ and $[H_{ikj}]$ are derived and listed in the appendix. For application to MSP synthesis, the time derivatives of equation (4.34) are obtained from the following:

$$\frac{d^n}{dt^n}\ [G_{ikj}]\ \{r_i\} + \frac{d^n}{dt^n}\ (S_{gj}[H_{ikj}])\ \{a_i\} = \frac{d^n}{dt^n}\ \{P_{ij}\} \qquad (4.35)$$

$$i = 1, 2, 3$$

$$j = 2, 3$$

Because of the limit to the number of positions available for synthesis (in this case, 3), the maximum value n can have is two. There are also four cases of MSP specifications: i) P-P-P, ii) P-PV or PV-P and iii) PVA. This allows equation (4.35) to be written for n = 1 at either j = 2 or j = 3 position. The case of n = 2 in equation (4.35) can only be written for the j = 3 position. Equation (4.35) for n = 1 is

$$[G_{ikj}^{1}]\ \{r_i\} + (S_{gj}^{1}[H_{ikj}] + S_{gj}[H_{ikj}^{1}])\ \{a_i\} = \{P_i^{\ 1}\} \qquad (4.36)$$

$$i = 1, 2, 3$$

$$j = 2 \text{ or } 3$$

and for n = 2, equation (4.35) is

$$[G_{ikj}^{2}]\ \{r_i\} + (S_{gj}^{2}[H_{ikj}] + 2S_{gj}^{1}[H_{ikj}^{1}])\ \{a_i\} = \{P_i^{\ 2}\} \qquad (4.37)$$

$$i = 1, 2, 3$$

$$j = 2 \text{ or } 3$$

Note that $S_{gj}^{n} = 0$ when equation (4.36) is written for j = 2. This shows that, in any case, there will only be two S_{gj}^{n}'s that will appear in the synthesis equations. The worst case would be the P-PV and PVA MSP specifications wherein one S_{gj}^{n} will appear in all the six equations. For the P-PV case, it will be S_{gj}, while for the PVA case, it would be S_{gj}^{n}. The expressions for the elements of matrix $[G_{ikj}^{n}]$ and $[H_{ikj}^{n}]$ are listed in the appendix for n = 0, n = 1 and n = 2.

CHAPTER 5

SOLUTIONS OF THE DIFFERENT SYNTHESIS CASES

This chapter will present the solutions to the different synthesis cases that can be considered for each of the four mechanisms: R-Sp-R, R-Sp-P, R-Cp-C and R-Sg-C. These different synthesis cases are determined by the number of positions specified and by the choice of which parameters are specified or considered as unknowns. Different synthesis cases are presented and the solution procedures for representative and unique cases will be shown. The solutions presented will consist of the intermediate equations required and the final system of equations to be solved. The solutions are assumed to be complete whenever a system of n non-homogeneous linear equations in n unknowns is obtained.

As mentioned in Chapter 3, the number of positions for synthesis can be from two to six for the R-Sp-R mechanism and either two or three for the R-Sp-P, the R-Cp-C and the R-Sg-C mechanisms. When less than the maximum number of positions are considered, only a finite number of parameters can be considered unknowns from the set of design parameters, the rest must be specified.

Reviewing the design parameters for the mechanism, they are, the three coordinates r_1, r_2, r_3 of vector $\underset{\sim}{R}$ and d_1 and d_2 of vector $\underset{\sim}{A}$ for the R-Sp-R mechanism, the two parameters d_1 and d_2 of vector $\underset{\sim}{A}$ for the R-Sp-P mechanism, any two coordinates of $\underset{\sim}{R}$, d_1 and d_2 of $\underset{\sim}{A}$, and e_1, e_2 of vector $\underset{\sim}{C}$ for the R-Cp-C mechanism, and the three coordinates of $\underset{\sim}{R}$ and parameters d_1 and d_2 of $\underset{\sim}{A}$ for the R-Sg-C mechanism.

The choice of the number of positions is in general determined by the required precision of the motion to be generated. In contrast, the choice of which parameters to treat as unknowns is not determined by the motion requirements. It can be arbitrary, or it can also be determined by additional mechanism constraints such as size, location of the intermediate pairs and orientation of pair elements. In some cases, the unknown parameters can be the parameters of one vector. This helps in visualizing the resulting solutions. Another important consideration in the choice of which parameters are to be specified and which are to be considered unknowns is the complexity of the solution procedure. It will be shown in the results that different choices of unknowns for the same number of positions can have varying degrees of complexity.

The following sections will list the synthesis cases considered for each of the four function generators, namely the R-Sp-R, R-Sp-P, R-Cp-C and R-Sg-C mechanisms.

5.1 Synthesis Cases for the R-Sp-R Mechanism

Table 5-1 shows the nine possible cases that can be considered for the R-Sp-R mechanism. The design parameters of the mechanism are the three coordinates r_1, r_2, r_3 of vector $\underset{\sim}{R}$, and the parameters d_1 and d_2 of vector $\underset{\sim}{A}$. Note that each case can have further sub-cases depending on specific combinations of the parameters considered as unknowns.

5.2 Synthesis Cases for the R-Sp-P Mechanism

The design parameters for this mechanism are the two unit vector parameters of vector $\underset{\sim}{A}$. One of these parameters is specified arbitrarily when considering the case of two-position synthesis.

TABLE 5-1

SYNTHESIS CASES FOR THE R-Sp-R MECHANISM

Two-Position Synthesis

Case R2: Any one of the coordinates r_1, r_2, r_3 of $\underset{\sim}{R}$ and the parameters d_1 and d_2 of $\underset{\sim}{A}$ can be considered as the unknown.

Three-Position Synthesis

Case R3-1: The vector $\underset{\sim}{A}$ in d_1 and d_2 are the unknowns.

Case R3-2: Any two of the three coordinates of $\underset{\sim}{R}$ are the unknowns.

Case R3-3: One coordinate of $\underset{\sim}{R}$ and one parameter of $\underset{\sim}{A}$ are the unknowns.

Four-Position Synthesis

Case R4-1: The vector $\underset{\sim}{R}$ in r_1, r_2 and r_3 is the unknown.

Case R4-2: The vector $\underset{\sim}{A}$ in d_1 and d_2 and one r_i are the unknowns.

Case R4-3: Two coordinates of $\underset{\sim}{R}$ and one parameter of $\underset{\sim}{A}$ are unknowns.

Five-Position Synthesis

Case R5: The vector $\underset{\sim}{R}$ (r_1, r_2, r_3) and one parameter of $\underset{\sim}{A}$ are unknowns.

Six-Position Synthesis

Case R6: The vector $\underset{\sim}{R}$ (r_1, r_2, r_3) and vector $\underset{\sim}{A}$ (d_1, d_2) are the unknowns.

Note: In each case, the parameters not chosen as unknowns can be specified arbitrarily.

This gives two synthesis cases for the R-Sp-P mechanism--the two position problem where either d_1 or d_2 of vector $\underset{\sim}{A}$ are unknowns and the three position problem where both d_1 and d_2 of vector $\underset{\sim}{A}$ are the unknowns.

5.3 Synthesis Cases for the R-Cp-C Mechanism

The design parameters of the R-Cp-C mechanism are any two coordinates r_1, r_2, r_3 of vector $\underset{\sim}{R}$, d_1 and d_2 of vector $\underset{\sim}{A}$, and e_1, e_2 of vector $\underset{\sim}{C}$, For this mechanism, there are two separate equations that are written for each displaced position plus one initial position equation.

Table 5-2 lists the synthesis cases for the R-Cp-C mechanism.

5.4 Synthesis Cases for the R-Sg-C Mechanism

The design parameters for the R-Sg-C mechanism are r_1, r_2, r_3 of vector $\underset{\sim}{R}$ and d_1, d_2 of vector $\underset{\sim}{A}$. The parameters of unit vector $\underset{\sim}{A}$ are, however, expressed in terms of the coordinates a_1, a_2, a_3 in the synthesis equations. Table 5-3 shows the five synthesis cases for the R-Sg-C mechanism.

5.5 R-Sp-R Mechanism Synthesis Solutions

The MSP synthesis equation for the R-Sp-R mechanism is recalled from section 4.2 as

$$\begin{aligned} &[(1 + pd_2)K_{1j}^n + K_{2j}^n d_1]r_1 + [K_{3j}^n + K_{4j}^n d_1 + K_{5j}^n d_2]r_2 \\ &\quad + [K_{6j}^n + K_{7j}^n d_1 + K_{8j}^n d_2]r_3 + [(1 + pd_2)K_{2j}^n + qK_{1j}^n d_1] = 0 \qquad (5.1) \end{aligned}$$

$$j = 2, 3, \text{ up to } 6$$

$$n = 0 \text{ or } 1 \text{ or } 2\ldots$$

where $d_1 = a_2/a_1$, $d_2 = a_3/a_1$

TABLE 5-2

SYNTHESIS CASES FOR THE R-Cp-C MECHANISM

Two-Position Synthesis

Case C2-1: Vector $\underset{\sim}{C}$ (e_1, e_2) and one d_k (parameter of vector $\underset{\sim}{A}$).

Case C2-2: Vector $\underset{\sim}{A}$ (d_1, d_2) and one r_i (coordinate of vector $\underset{\sim}{R}$).

Case C2-3: Vector $\underset{\sim}{C}$ (e_1, e_2) and one r_i (coordinate of vector $\underset{\sim}{R}$).

Case C2-4: Vector $\underset{\sim}{A}$ (d_1, d_2) and one e_ℓ (parameter of vector $\underset{\sim}{C}$).

Three-Position Synthesis

Case C3-1: Vector $\underset{\sim}{A}$ in d_1, d_2, one e_ℓ and two r_i-s are the unknowns.

Case C3-2: Vector $\underset{\sim}{C}$ in e_1, e_2, one d_k and two r_i-s are the unknowns.

TABLE 5-3

SYNTHESIS CASES FOR THE R-Sg-C MECHANISM

Two-Position Synthesis

Case G2-1: Vector $\underset{\sim}{A}$ is the unknown.

Case G2-2: Two coordinates of $\underset{\sim}{R}$ are unknowns.

Case G2-3: One parameter each of vectors $\underset{\sim}{A}$ and $\underset{\sim}{R}$ are the unknowns.

Three-Position Synthesis

Case G3-1: Vector $\underset{\sim}{A}$ in d_1, d_2 and two coordinates of $\underset{\sim}{R}$ are unknowns.

Case G3-2: Vector $\underset{\sim}{R}$ in (r_1, r_2, r_3) and either d_1 or d_2 of $\underset{\sim}{A}$ are unknowns.

Note: In each case, the parameters not chosen as unknowns can be specified arbitrarily.

The solutions presented will begin with the three-position cases since the equation obtained for two-position synthesis is a linear equation with one unknown.

5.5.1 Solutions for the Three-Position Synthesis Cases

Case R3-1 and R3-2

The unknowns in this case are either the two parameters of A or the two coordinates of the vector $\underset{\sim}{R}$. Letting these unknowns be represented by x and y, the following system of two linear equations in two unknowns is obtained:

$$A_j x + B_j y + C_j = 0 \qquad (5.2)$$

$$j = 2, 3$$

where A_j, B_j and C_j are the appropriate coefficients obtained from the K_{ij}^{n}-s and the specified parameters.

Case R3-2

The unknowns in this case are one r_i and one d_k. From this, there can be six combinations of r_i $(i = 1, 2, 3)$ and d_k. The case of r_1 and d_1 as the unknowns will be shown. The solutions to the other combinations are obtained in a similar manner.

The synthesis equations obtained are arranged as

$$(A_j + B_j d_1) r_1 + (C_j + D_j d_1) = 0 \qquad (5.3)$$

$$j = 2, 3$$

The solution procedure for the system of equation (5.3) is shown in example 1 of Chapter 2.

5.5.2 Solutions for the Four-Position Synthesis Cases

Case R4-1

This is a case of four MSP synthesis that results in a linear system of equations. The unknowns in this case are the coordinates r_1, r_2 and r_3 of vector $\underset{\sim}{R}$, and the system of equations is obtained from equation (5.1).

Case R4-2 and R4-3

These two cases, where the unknowns are either two r_i-s and one d_k or two d_k-s and one r_i, will lead to the following system of equations:

$$(A_j + B_j w)u + (C_j + D_j w)v + (E_j + F_j w) = 0 \qquad (5.4)$$

$$j = 2, 3, 4$$

The variables u, v, w stand for the unknown r_i $(i = 1, 2 \text{ or } 3)$ and d_k $(k = 1 \text{ or } 2)$. The constants A_j, B_j, C_j, D_j, E_j and F_j are deterministic functions of the K_{ij}^n-s and the specified design parameters. Also, the variable w represents the one unknown d_k for Case R4-2 or the one unknown r_i of Case R4-3.

The solution procedure for the system of equations (5.4) is shown in example 1 of Chapter 2.

5.5.3 Solution for the Five-Position Synthesis Case

Case R5

The unknowns in this case are r_1, r_2, r_3 of vector $\underset{\sim}{R}$ and either d_1 or d_2 of vector $\underset{\sim}{A}$. From the synthesis equation of the R-Sp-R mechanism, the following system of equations is obtained for the case of d_1 as the fourth unknown:

$$(A_j + B_j d_1) r_1 + (C_j + D_j d_1) r_2 + (E_j + F_j d_1) r_3 + (G_j + H_j d_1) = 0 \quad (5.5)$$

$$j = 2, 3, 4, 5$$

where

$$\begin{aligned} A_j &= (1 + pd_2) K_{1j}^n \\ B_j &= K_{2j}^n \\ C_j &= K_{3j}^n + K_{5j}^n d_2 \\ D_j &= K_{4j}^n \\ E_j &= K_{6j}^n + K_{8j}^n d_2 \\ F_j &= K_{7j}^n \\ G_j &= (1 + pd_2) K_{2j}^n \\ H_j &= qK_{1j}^n \end{aligned} \quad (5.6)$$

The solutions to the system of equations (5.5) are obtained by first finding the roots of d_1 from the eliminant polynomial determined by

$$\begin{vmatrix} K_{1j}^n & (C_j + D_j d_1) & (E_j + F_j d_1) & K_{2j}^n \end{vmatrix} = 0 \quad (5.7)$$

$$j = 2, 3, 4, 5$$

The eliminant is only quadratic because the dependence of the A_j, B_j, G_j and H_j terms allow a constant polynomial independent of motion specifications to be factored out. The zero or two real roots of the resulting quadratic in d_1 are then substituted into any three equations of (5.5) to obtain r_1, r_2 and r_3.

5.5.4 Solution for the Six-Position Synthesis Case

Case R6

The unknowns in this case are all the design parameters r_1, r_2, r_3 of vector $\underset{\sim}{R}$ and d_1 and d_2 of vector $\underset{\sim}{A}$.

A direct elimination of the design parameters from the set of equations obtained will result in a null polynomial for the eliminant. This is due to the relationships among some of the coefficients of the variables in the equation. To obtain an eliminant polynomial, the coefficients are regrouped and auxiliary variables are introduced. Thus, the synthesis equations from equation (5.1) are arranged so that all the coefficients are independent and in terms of one variable d_2.

$$(K_{3j}^{n} + K_{5j}^{n}d_2)r_2 + (K_{6j}^{n} + K_{8j}^{n}d_2)r_3 + K_{1j}^{n}x + K_{2j}^{n}y + K_{4j}^{n}d_1r_2 + K_{7j}^{n}d_1r_3 = 0 \qquad j = 2, 3, 4, 5, 6 \qquad (5.8)$$

where the auxiliary variables x and y are

$$x = (1 + pd_2)r_1 + qd_1$$

$$y = (1 + pd_2) + d_1r_1$$

Equation (5.8) is considered to have unknowns r_2, r_3, x, y, d_1r_2, and d_1r_3 in five equations. Multiplying the five equations of (5.8) by d_1 will introduce four additional unknowns d_1x, d_1y, $d_1{}^2r_2$ and $d_1{}^2r_3$. Thus, the eliminant in d_2 is obtained from the following 10 x 10 determinant set to zero.

$$\begin{vmatrix} (K_{3j}^{n} + K_{5j}^{n}d_2) & (K_{6j}^{n} + K_{8j}^{n}d_2) & K_{1j}^{n} & K_{2j}^{n} & K_{4j}^{n} & K_{7j}^{n} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (K_{3j}^{n} + K_{5j}^{n}d_2) & (K_{6j}^{n} + K_{8j}^{n}d_2) & K_{1j}^{n} & K_{2j}^{n} & K_{4j}^{n} & K_{7j}^{n} \end{vmatrix} = 0 \qquad (5.9)$$

$$j = 2, 3, 4, 5, 6$$

The eliminant above is determined to be cubic by using the method of Laplace's Development (see Appendix A). This will give one or three real roots for d_2. These roots for d_2 are the values of d_2 that are common to the five synthesis equations.

The next step now is to obtain the values of d_1 that will also be common to all the five synthesis equations. One can use the values of d_2 and substitute them into any four of the five equations and then get an eliminant in d_1 (see the Five-Position Case). This method will however introduce an extraneous root. One of the roots of the resulting quadratic in d_1, using the five-position procedure, will satisfy only the five positions from which the four equations were obtained. The other root for d_1 would be the correct value which will eventually satisfy the six prescribed positions. Rather than determining the values of d_1 by this method--and just checking the results later for the correct root--a better method would be to set up an eliminant in d_1 which will give only the one correct d_1.

This can be achieved by using all of the five synthesis equations at the same time in the elimination procedure.

The synthesis equations (d_2 already known) can be arranged in the following forms.

$$[(1 + pd_2)K_{1j}^{n} + K_{2j}^{n}d_1]r_1 + [(K_{3j}^{n} + K_{5j}^{n}d_2) + K_{4j}^{n}d_1]r_2 + (K_{6j}^{n} + K_{8j}^{n}d_2)r_3 + K_{7j}^{n}d_1r_3 + [(1 + pd_2)K_{2j}^{n} + qK_{1j}^{n}d_1] = 0$$

or, using the auxiliary variables x and y, (5.10)

$$K_{1j}^{n}x + [(K_{3j}^{n} + K_{5j}^{n}d_2) + K_{4j}^{n}d_1]r_2 + (K_{6j}^{n} + K_{8j}^{n}d_2)r_3 + K_{7j}^{n}d_1r_3 + K_{2j}^{n}y = 0$$

Although the two equations above are expressed in different variables, the resulting eliminant will be identical and is

$$\left| \; K_{1j}^{n} \quad [(K_{3j}^{n} + K_{5j}^{n}d_2) + K_{4j}^{n}d_1] \quad (K_{6j}^{n} + K_{8j}^{n}d_2) \quad K_{7j}^{n} \quad K_{2j}^{n} \; \right| = 0 \qquad (5.11)$$

$$j = 2, 3, 4, 5, 6$$

Equation (5.11) is readily obtained from the second equation of (5.10). The first equation of (5.10) is shown because from it, one can show that its eliminant consists of equation (5.11) multiplied by the constant polynomial $[(1 + pd_2)^2 - qd_1^{\,2}]$. The presence of this polynomial factor explains why using dyalitic elimination directly on the design parameters of the five synthesis equations will always result into a null polynomial. This is because the condition of having common roots is already satisfied.

The value of d_1 obtained from the linear eliminant of (5.11) and the corresponding value of d_2 are then substituted back into any three of the five synthesis equations to get r_1, r_2 and r_3.

The K_{ij}^{n}-s in general have absolute values less than one so that determinants of high order like five or ten will give very small numbers. This is remedied by simply normalizing the resulting polynomials or by multiplying the K_{1j}^{n}-s by a convenient number before evaluating determinants.

5.6 R-Sp-P Mechanism Synthesis Solutions

The solution procedures for the two synthesis cases of the R-Sp-P are not shown since the two-position problem will result in a linear equation in one unknown and the three-position case also results in a linear system of two equations in two unknowns.

5.7 R-Cp-C Mechanism Synthesis Solutions

The synthesis equations for the R-Cp-C mechanism are recalled from section 4.4 as

$$e_1d_1 + e_2d_2 + 1 = 0 \tag{5.12}$$

$$L_{1j}^{n} + L_{2j}^{n}e_1 + L_{3j}^{n}e_2 + L_{4j}^{n}d_1 + L_{5j}^{n}d_1e_1 + L_{6j}^{n}d_1e_2 + L_{7j}^{n}d_2 + L_{8j}^{n}d_2e_1 + L_{9j}^{n}d_2e_2 = 0 \tag{5.13}$$

$$j = 2, 3$$

and

$$[(1 + pd_2)M_{1j}^{n} + M_{2j}^{n}d_1]r_1 + [M_{3j}^{n} + M_{4j}^{n}d_1 + M_{5j}^{n}d_2]r_2 + [M_{6j}^{n} + M_{7j}^{n}d_1 + M_{8j}^{n}d_2]r_3 + [M_{9j}^{n} + qM_{1j}^{n}d_1 + M_{10j}^{n}d_2] = 0 \tag{5.14}$$

$$j = 2, 3$$

where d_1 and d_2 are the parameters of vector $\underset{\sim}{A}$

e_1 and e_2 are the parameters of vector $\underset{\sim}{C}$

and r_1, r_2 and r_3 are the coordinates of vector $\underset{\sim}{R}$.

5.7.1 Solutions for the Two-Position Synthesis Cases

Case C2-1

The unknowns in this case are e_1, e_2 of vector $\underset{\sim}{C}$ and one d_k. The case of d_1 as the third unknown will be considered. Deleting the j = 2 subscript in the coefficients, the synthesis equations obtained are written as

$$e_1d_1 + e_2d_2 + 1 = 0 \tag{5.15}$$

$$(A + Bd_1)e_1 + (C + Dd_1)e_2 + (E + Fd_1) = 0 \tag{5.16}$$

and

$$Gd_1 + H = 0 \tag{5.17}$$

The coefficients A, B ...G, H are constants that are obtained from the L_{ij}^n-s, M_{ij}^n-s and the specified design parameters.

The solutions to e_1, e_2 and d_1 are obtained by first solving for d_1 from equation (5.17) and then substituting this value of d_1 into equations (5.15) and (5.16). This will result in a system of two linear equations in unknowns e_1 and e_2.

Case C2-2

The unknowns for this case are d_1, d_2 and one r_i. The case of r_1 as the third unknown will be shown. The other two cases are solved in the same manner.

The resulting system of equations is rearranged as

$$\begin{aligned} e_1d_1 + e_2d_2 + 1 &= 0 \\ Ad_1 + Bd_2 + C &= 0 \qquad (5.18) \\ \text{and} \quad (D + Ed_1 + Fd_2)r_1 + (G + Hd_1 + Id_2) &= 0 \end{aligned}$$

where A, B, C ....H, I are obtained from the L_{ij}^n-s, M_{ij}^n-s and the specified parameters.

The solutions for d_1, d_2 and r_1 are obtained by first solving for d_1 and d_2 from the first two equations which are linear in the two unknowns, and then using the resulting values of d_1 and d_2 in the third equation to get r_1.

Case C2-3

The unknowns in this case are e_1, e_2 and one r_i. The resulting system of equations is similar to that of case C2-1, and is solved in a similar manner.

Case C2-4

This is the only case of two-position synthesis that results in non-linear equations. The unknowns in this case are d_1, d_2 and one e_i. The case of e_1 as the third unknown will be considered. The system of equations obtained is

$$e_1 d_1 + e_2 d_2 + 1 = 0 \tag{5.19}$$

$$(A + Be_1)d_1 + (C + De_1)d_2 + Fe_1 = 0 \tag{5.20}$$

and

$$Gd_1 + Gd_2 + 1 = 0 \tag{5.21}$$

where A, B, C....G, H are constants that are obtained from the appropriate L_{ij}^n-s, M_{ij}^n-s and the specified parameters.

Solutions to e_1 are obtained from the eliminant derived from equations (5.19) to (5.21) expressed as the determinant

$$\begin{vmatrix} e_1 & e_2 & 1 \\ (A + Be_1) & (C + De_1) & (E + Fe_1) \\ G & H & I \end{vmatrix} = 0 \qquad (5.22)$$

The real roots (if they exist) of the quadratic in e_1 are then substituted into any two equations of (5.19) to (5.21) to obtain the values of d_1 and d_2 corresponding to each solution of e_1.

5.7.2 Solutions for the Three-Position Synthesis Cases

Case C3-1

The unknowns in this case are d_1, d_2, two r_i-s and either e_1 or e_2. The case of e_1, r_1, r_2, d_1 and d_2 as the unknowns will be considered. The other combinations of e_1 or e_2 and two r_i-s are solved in a similar manner.

The following equations are obtained from the synthesis equations:

$$e_1 d_1 + e_2 d_2 + 1 = 0 \qquad (5.23)$$

$$(A_j + B_j e_1)d_1 + (C_j + D_j e_1)d_2 + (E_j + F_j e_1) = 0 \qquad (5.24)$$

$$j = 2, 3$$

and

$$(G_j + H_j d_1 + I_j d_2)r_1 + (J_j + K_j d_1 + L_j d_2)r_2$$

$$+ (M_j + N_j d_1 + O_j d_2) = 0 \qquad j = 2, 3 \qquad (5.25)$$

where the A_j, B_j, C_j ....O_j coefficients are obtained from the L_{ij}^n-s, M_{ij}^n-s and the specified parameters.

The system of five equations can be solved by considering the three equations of (5.23) and (5.24) separately from the two equations of (5.25). In the first set, the solutions for e_1 are obtained from the eliminant, expressed as the determinant

$$\begin{vmatrix} e_1 & e_2 & 1 \\ (A_2 + B_2e_1) & (C_2 + D_2e_1) & (E_2 + F_2e_1) \\ (A_3 + B_3e_1) & (C_3 + D_3e_1) & (E_3 + F_3e_1) \end{vmatrix} = 0 \qquad (5.26)$$

The one or three real roots of e_1 from the third degree eliminant polynomial are then substituted into any two equations of (5.23) and (5.24) to get d_1 and d_2. With d_1 and d_2 obtained, r_1 and r_2 are solved for from the two equations of (5.25).

Case C3-2

The unknowns for this case are e_1, e_2, one d_k and two r_i-s. The resulting system of equations is similar to that of case C3-1, and the solution procedure is the same as for that case.

5.8 R-Sg-C Mechanism Synthesis Solutions

The solutions for the synthesis of the R-Sg-C mechanism require a different approach because the form of the synthesis equations differs from those of the R-Sp-R and R-Cp-C mechanisms. Although the terms d_1 and d_2 do not appear in the equations, the solutions will be finally based on these two parameters of $\underset{\sim}{A}$.

5.8.1 Solutions for the Two-Position Synthesis Cases

For the two-position synthesis, the two possible cases of MSP specification are either P-P or PV. Both cases will give the following equation:

$$[G^n_{ik}]\ \{r_i\} + S^n_g[H^n_{ik}]\ \{a_i\} = \{P^n_i\} \tag{5.27}$$

$$i = 1, 2, 3$$

$$k = 1, 2, 3$$

where $n = 0$, for displacement specifications on the second position

or $n = 1$, when the second MSP is of first order.

Case G2-1

The unknowns in this case are the two parameters d_1 and d_2 of vector $\underset{\sim}{A}$. The synthesis equations can be arranged as

$$a_1S_g(A_i + B_id_1 + C_id_2) = D_i \tag{5.28}$$

$$i = 1, 2, 3$$

Equation (5.28) is solved for d_1, d_2 and S_g by first eliminating a_1S_g from the first two and last two equations of (5.28). This will give two linear equations in the unknowns d_1 and d_2. With d_1 and d_2 known, a_1 can be determined and then S_g is obtained from one of the equations of (5.28).

Case G2-2

The case of considering two r_i-s as unknowns will result in a system of three linear equations in three unknowns. With r_1 and r_2 as the unknowns, the equation obtained is

$$A_i r_1 + B_i r_2 + C_i S_g = D_i \qquad (5.29)$$

$$i = 1, 2, 3$$

Case G2-3

The unknowns in this case are one parameter of $\underset{\sim}{A}$, one coordinate of the vector $\underset{\sim}{R}$ and S_g. The case of d_1 and r_1 as the unknown parameters will be considered. The other cases will require the same solution procedure.

The synthesis equations obtained are rearranged and simplified as

$$A_i r_1 + (B_i + D_i d_1) a_1 S_g = E_i \qquad (5.30)$$

$$i = 1, 2, 3$$

Eliminating r_1 from the first two equations and also from the last two equations will give two equations of the following form:

$$\begin{aligned} a_1 S_g (A' + B' d_1) &= C' \\ a_1 S_g (D' + E' d_1) &= F' \end{aligned} \qquad (5.31)$$

Dividing the first equation by the second equation in (5.31) will result in a quadratic in d_1 whose roots are the solutions for d_1.

With d_1 known, and d_2 previously specified, the values of a_1, a_2 and a_3 can be obtained. The values of a_1 and d_1 are then substituted into any two equations of (5.30), resulting in a system of two linear equations in the unknowns r_1 and S_g.

5.8.2 Solutions for the Three-Position Synthesis Cases

The synthesis equations obtained for three-position synthesis have different forms which depend on whether only discrete positions are considered or a combination of discrete and infinitesimally close positions.

For discrete positions P-P-P and PV-P MSP, the synthesis equations have the form

$$[G_{ik2}]\ \{r_i\} + S_{g2}^{n}[H_{ik2}]\ \{a_i\} = \{P_{i2}^{n}\}$$

and (5.32)

$$[G_{ik3}]\ \{r_i\} + S_{g3}[H_{ik3}]\ \{a_i\} = \{P_{i3}\}$$

where $n = 0$ for P-P-P

$n = 1$ for PV-P

For the three MSP specifications of P-PV and PVA, the synthesis equations have the following forms:

$$[G_{ik2}]\ \{r_i\} + S_{g2}^{a}[H_{ik2}^{a}]\ \{a_i\} = \{P_{i2}^{a}\} \qquad (5.33)$$

$i = 1, 2, 3$

$k = 1, 2, 3$

and

$$[G_{ik3}^{b}]\ \{r_i\} + S_{g3}^{b}[H_{ik3}] + S_{g2}^{a}[H_{ik3}^{a}]\ \{a_i\} = \{P_{i3}^{b}\} \qquad (5.34)$$

$i = 1, 2, 3$

$k = 1, 2, 3$

where $a = 0$ and $b = 1$ for P-PV MSP

$a = 1$ and $b = 2$ for PVA MSP

Case G3-1

The unknowns in this case are two r_i-s of $\underset{\sim}{R}$ and the vector $\underset{\sim}{A}$ (d , d_2). The case of r_1 and r_2 as the unknowns and the P-PV MSP specification will be considered. The procedure is the same for different combinations of unknown r_i-s and MSP specifications.

The synthesis equations obtained are

$$J_{i2} + A_{i2}r_1 + B_{i2}r_2 + [(C_{i2} + D_{i2}d_1) + E_{i2}d_2]a_1 S_{g2} = 0 \qquad i = 1, 2, 3 \tag{5.35}$$

and

$$J_{i3} + A_{i3}r_1 + B_{i3}r_2 + [(C_{i2} + D_{i2}d_1) + E_{i2}d_2]a_1 S^1_{g3} + [(F_{i3} + G_{i3}d_1) + H_{i3}d_2]a_1 S_{g2} = 0 \qquad i = 1, 2, 3 \tag{5.36}$$

where the A_{ij}-s, B_{ij}-s, ... are the corresponding coefficients of the appropriate parameters and are determined from the specified parameters and the elements of $[G^n_{ikj}]$ and $[H^n_{ikj}]$.

Making d_1 a coefficient variable, the system of six equations of (5.35) and (5.36) is considered to have unknowns 1, r_1, r_2, $a_1 S_{g2}$, $a_1 S^1_{g3}$, $d_2 a_1 S_{g2}$ and $d_2 a_1 S^1_{g3}$. The method of elimination shown in example 2 of Chapter 2 is used to solve the system of equations obtained from (5.35) and (5.36). The system of equations of (5.35) and (5.36) are multiplied by d_2 to obtain the following matrix equation:

$$\begin{bmatrix} [\,M_2\,]_{3x5} & [\,0\,]_{3x7} \\ [\,M_3\,]_{3x7} & [\,0\,]_{3x5} \\ \multirow{2}{*}{$\begin{bmatrix} 0 \end{bmatrix}_{6x4}$} & [\,N_2\,]_{3x8} \\ & [\,N_3\,]_{3x8} \end{bmatrix}_{12x12} \begin{Bmatrix} 1 \\ r_1 \\ r_2 \\ \vdots \\ d_2^{\,2} a_1 S^1_{g3} \end{Bmatrix} = 0 \tag{5.37}$$

where the 3 x 5 matrix $[M_2]$ is

$$[\ J_{i2} \quad A_{i2} \quad B_{i2} \quad (C_{i2} + D_{i2}d_1) \quad E_{i2}\ ] \qquad (5.38)$$

$$i = 1, 2, 3$$

the 3 x 7 matrix $[M_3]$ is

$$[\ J_{i3} \quad A_{i3} \quad B_{i3} \quad (F_{i3} + G_{i3}d_1) \quad H_{i3} \quad (C_{i3} + D_{i3}d_1) \quad E_{i3}]$$

$$i = 1, 2, 3 \qquad (5.39)$$

the 3 x 8 matrix $(N_2]$ is

$$[(C_{i2} + D_{i2}d_1) \quad 0 \quad 0 \quad I_{i2} \quad A_{i2} \quad B_{i2} \quad E_{i2} \quad 0] \qquad (5.40)$$

$$i = 1, 2, 3$$

and the 3 x 8 matrix $(N_3]$ is

$$[(F_{i3} + G_{i3}d_1) \quad 0 \quad (C_{i2} + D_{i2}d_1) \quad I_{i3} \quad A_{i3} \quad B_{i3} \quad H_{i3} \quad E_{i3}\ ]$$

$$i = 1, 2, 3 \qquad (5.41)$$

Setting the determinant of the matrix in equation (5.73) to zero will give a cubic eliminant polynomial in d_1, the roots of which are the solutions to d_1.

The roots of d_1 are then substituted into any eleven equations of (5.37). This will result in a system of eleven non-homogeneous linear equations in eleven unknowns.

Case G3-2

The unknowns in this case are r_1, r_2, r_3 and d_1 or d_2. Using d_1 as the unknown and the case of P-PV MSP (since it represents the form whose solution procedure is most general), the synthesis equations obtained are

$$A_{i2}r_1 + B_{i2}r_2 + C_{i2}r_3 + (D_{i2} + E_{i2}d_1)a_1S_{g2} + H_{i2} = 0 \qquad (5.42)$$

$$i = 1, 2, 3$$

and

$$A_{i3} + B_1 + B_{i3}r_2 + C_{i3}r_3 + (D_{i2} + E_{i2}d_1)a_1S_{g3}^{1} + (F_{i3} + G_{i3}d_1)a_1S_{g2} + H_{i3} = 0$$

$$i = 1, 2, 3 \qquad (5.43)$$

Making the variables r_1, r_2, r_3, a_1S_{g2}, a_1S_{g3} and 1 the unknowns gives the eliminant polynomial in the following 6 x 6 determinant.

$$\begin{vmatrix} A_{i2} & B_{i2} & C_{i2} & (D_{i2} + E_{i2}d_1) & 0 & H_{i2} \\ A_{i3} & B_{i3} & C_{i3} & (F_{i3} + G_{i3}d_1) & (D_{i2} + E_{i2}d_1) & H_{i3} \end{vmatrix} = 0 \qquad (5.44)$$

$$i = 1, 2, 3$$

The roots of the quadratic eliminant in d_1 are then substituted into any five equations of (5.42) and (5.43). From the resulting linear system, the remaining five unknowns r_1, r_2, r_3, S_{g2} and S_{g3}^{1} are then obtained.

CHAPTER 6

KINEMATIC ANALYSIS OF THE THREE-LINK FUNCTION GENERATORS

In this chapter, the displacement, velocity and acceleration analysis will be carried out for the four function generators namely- the R-Sp-R, R-Sp-P, R-Cp-C and R-Sg-C. The transmission characteristics of these mechanisms will also be evaluated.

Sandor, Kohli, Hernandez and Ghosal [41] have shown the analysis results for the R-Sp-R, R-Sp-P and R-Sg-C mechanisms. The same method will be used here for the R-Cp-C mechanism.

The geometry and motion of the mechanisms considered are described by common equations. These equations, obtained from equations (3.11), (3.14), (3.15) and (3.18), together with their derivatives are listed for substitution to the appropriate equations of motion.

$$\underset{\sim}{R}_j - \underset{\sim}{R}'_j = \underset{\sim}{S}_j - C_1\phi_j\underset{\sim}{U}_{BL} - S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L}) - \underset{\sim}{u}_B S_{cj} \tag{6.1}$$

$$\dot{\underset{\sim}{R}}_j - \dot{\underset{\sim}{R}}'_j = \underset{\sim}{M}_{Aj}\dot{\theta}_j - \underset{\sim}{M}_{Bj}\dot{\phi}_j - \underset{\sim}{u}_B\dot{S}_{cj} \tag{6.2}$$

$$\ddot{\underset{\sim}{R}}_j - \ddot{\underset{\sim}{R}}'_j = \underset{\sim}{N}_{Aj}\dot{\theta}_j^2 + \underset{\sim}{M}_{Aj}\ddot{\theta}_j - \underset{\sim}{N}_{Bj}\dot{\phi}_j^2 - \underset{\sim}{M}_{Bj}\ddot{\phi}_j - \underset{\sim}{u}_B\ddot{S}_{cj} \tag{6.3}$$

$$\underset{\sim}{A}_j = \underset{\sim}{A} + C_1\phi_j\underset{\sim}{U}_{BA} + S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A}) \tag{6.4}$$

$$\dot{\underset{\sim}{A}}_j = \underset{\sim}{V}_{Bj}\dot{\phi}_j \tag{6.5}$$

$$\ddot{\underset{\sim}{A}}_j = \underset{\sim}{W}_{Bj}\dot{\phi}_j^2 + \underset{\sim}{V}_{Bj}\ddot{\phi}_j \tag{6.6}$$

$$\underset{\sim}{C}_j = \underset{\sim}{C} + C_1\theta_j\underset{\sim}{U}_{AC} + S\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{C}) \tag{6.7}$$

$$\dot{\underset{\sim}{C}}_j = \underset{\sim}{V}_{Aj}\dot{\theta}_j \tag{6.8}$$

$$\ddot{\underset{\sim}{C}}_j = \underset{\sim}{W}_{Aj}\dot{\theta}_j^{2} + \underset{\sim}{V}_{Aj}\ddot{\theta}_j \tag{6.9}$$

where $$\underset{\sim}{S}_j = C_1\theta_j\underset{\sim}{U}_{AR} + S\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) \tag{6.10}$$

$$\underset{\sim}{M}_{Aj} = \cos\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) - \sin\theta_j\underset{\sim}{U}_{AR} \tag{6.11}$$

$$\underset{\sim}{M}_{Bj} = \cos\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L}) - \sin\phi_j\underset{\sim}{U}_{BL} \tag{6.12}$$

$$\underset{\sim}{N}_{Aj} = -\sin\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) - \cos\theta_j\underset{\sim}{U}_{AR} \tag{6.13}$$

$$\underset{\sim}{N}_{Bj} = -\sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L}) - \cos\phi_j\underset{\sim}{U}_{BL} \tag{6.14}$$

$$\underset{\sim}{V}_{Bj} = \cos\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A}) - \sin\phi_j\underset{\sim}{U}_{BA} \tag{6.15}$$

$$\underset{\sim}{V}_{Aj} = \cos\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{C}) - \sin\theta_j\underset{\sim}{U}_{AC} \tag{6.16}$$

$$\underset{\sim}{W}_{Bj} = -\sin\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A}) - \cos\phi_j\underset{\sim}{U}_{BA} \tag{6.17}$$

$$\underset{\sim}{W}_{Aj} = -\sin\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{C}) - \cos\theta_j\underset{\sim}{U}_{AC} \tag{6.18}$$

6.1 Analysis of the R-Sp-R Mechanism

The constraint equation and its time derivatives for the R-Sp-R mechanism are

$$(\underset{\sim}{R}_j - \underset{\sim}{R}_j') \cdot \underset{\sim}{A}_j = 0 \tag{6.19}$$

$$(\dot{\underset{\sim}{R}}_j - \dot{\underset{\sim}{R}}_j') \cdot \underset{\sim}{A}_j + (\underset{\sim}{R}_j - \underset{\sim}{R}_j') \cdot \dot{\underset{\sim}{A}}_j = 0 \tag{6.20}$$

and $$(\ddot{\underset{\sim}{R}}_j - \ddot{\underset{\sim}{R}}_j') \cdot \underset{\sim}{A}_j + 2(\dot{\underset{\sim}{R}}_j - \dot{\underset{\sim}{R}}_j') \cdot \dot{\underset{\sim}{A}}_j + (\underset{\sim}{R}_j - \underset{\sim}{R}_j') \cdot \ddot{\underset{\sim}{A}}_j = 0 \tag{6.21}$$

Substituting equations (6.1), (6.2) and (6.3) with $S_{Cj} = 0$, and equations (6.4) to (6.6) and simplifying the resulting expressions will give the following output motion parameters ϕ_j, $\dot{\phi}_j$, $\ddot{\phi}_j$ in terms of the input motion parameters θ_j, $\dot{\theta}_j$ and $\ddot{\theta}_j$. The displacement relation obtained is

$$\cos\phi_j[(\underset{\sim}{U}_{BL} + \underset{\sim}{S}_j) \cdot \underset{\sim}{U}_{BA}] + \sin\phi_j[(\underset{\sim}{L} + \underset{\sim}{S}_j) \cdot (\underset{\sim}{u}_B \times \underset{\sim}{A})] + \underset{\sim}{S}_j \cdot \underset{\sim}{A} - \underset{\sim}{U}_{BA} \cdot (\underset{\sim}{U}_{BL} + \underset{\sim}{S}_j) = 0 \qquad (6.22)$$

Substituting the following half angle identities

$$\cos\alpha = [1 - \tan^2(\alpha/2)] / [1 + \tan^2(\alpha/2)] \qquad (6.23)$$

and $$\sin\alpha = 2\tan(\alpha/2) / [1 + \tan^2(\alpha/2)] \qquad (6.24)$$

and simplifying the resulting quadratic equation yields

$$\tan(\phi_j/2) = (-b \pm \sqrt{a^2 + b^2 - c^2}) / (c - a) \qquad (6.25)$$

where
$$a = (\underset{\sim}{U}_{BL} + \underset{\sim}{S}_j) \cdot \underset{\sim}{U}_{BA}$$
$$b = (\underset{\sim}{L} + \underset{\sim}{S}_j) \cdot (\underset{\sim}{u}_B \times \underset{\sim}{A}) \qquad (6.26)$$
$$c = \underset{\sim}{S}_j \cdot \underset{\sim}{A} - \underset{\sim}{U}_{BA} \cdot (\underset{\sim}{U}_{BL} + \underset{\sim}{S}_j)$$

With ϕ_j known, the velocity and acceleration are obtained from the following equations:

$$\dot{\phi}_j = [(\underset{\sim}{M}_{Aj} \cdot \underset{\sim}{A}_j) / (\underset{\sim}{M}_{Bj} \cdot \underset{\sim}{A}_j - (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \underset{\sim}{V}_{Bj})]\,\dot{\theta}_j \qquad (6.27)$$

$$\ddot{\phi}_j = [(\underset{\sim}{N}_{Aj} \cdot \underset{\sim}{A}_j)\,\dot{\theta}_j^2 + (\underset{\sim}{M}_{Aj} \cdot \underset{\sim}{A}_j)\ddot{\theta}_j + 2(\underset{\sim}{M}_{Aj} \cdot \underset{\sim}{V}_{Bj})\dot{\theta}_j\dot{\phi}_j + (-\underset{\sim}{N}_{Bj} \cdot \underset{\sim}{A}_j - 2\underset{\sim}{M}_{Bj} \cdot \underset{\sim}{V}_{Bj} + (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \underset{\sim}{W}_j)\dot{\phi}_j^{\,2}] / D \qquad (6.28)$$

where $$D = \underset{\sim}{M}_{Bj} \cdot \underset{\sim}{A}_j - (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \underset{\sim}{V}_{Bj} \qquad (6.29)$$

6.2 Analysis of the R-Sp-P Mechanism

The displacement, velocity and acceleration equations for the R-Sp-P mechanism are obtained using equations (6.1) to (6.6) in equations (6.19) to (6.21) with the terms ϕ_j, $\dot{\phi}_j$ and $\ddot{\phi}_j$ set to zero. The results are

$$S_{cj} = \{[C_1\theta_j \underset{\sim}{U}_{AR} + S\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R})] \cdot \underset{\sim}{A}\} / \underset{\sim}{u}_B \cdot \underset{\sim}{A} \qquad (6.30)$$

$$\dot{S}_{cj} = [(\underset{\sim}{M}_{Aj} \cdot \underset{\sim}{A}) / \underset{\sim}{u}_B \cdot \underset{\sim}{A}]\, \dot{\theta}_j \qquad (6.31)$$

$$\ddot{S}_{cj} = [(\underset{\sim}{N}_{Aj} \cdot \underset{\sim}{A})\dot{\theta}_j^{\,2} + (\underset{\sim}{M}_{Aj} \cdot \underset{\sim}{A})\, \ddot{\theta}_j] / \underset{\sim}{u}_B \cdot \underset{\sim}{A} \qquad (6.32)$$

6.3 Analysis of the R-Cp-C Mechanism

The constraint equations for the R-Cp-C mechanism are listed for displacements,

$$\underset{\sim}{A}_j \cdot \underset{\sim}{C}_j = 0 \qquad (6.33)$$

$$(\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \underset{\sim}{A}_j = 0 \qquad (6.34)$$

for velocities,

$$\dot{\underset{\sim}{A}}_j \cdot \underset{\sim}{C}_j + \underset{\sim}{A}_j \cdot \dot{\underset{\sim}{C}}_j = 0 \qquad (6.35)$$

$$(\dot{\underset{\sim}{R}}_j - \dot{\underset{\sim}{R}}'_j) \cdot \underset{\sim}{A}_j + (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \dot{\underset{\sim}{A}}_j = 0 \qquad (6.36)$$

and for acceleration,

$$\ddot{\underset{\sim}{A}}_j \cdot \underset{\sim}{C}_j + 2\dot{\underset{\sim}{A}}_j \cdot \dot{\underset{\sim}{C}}_j + \underset{\sim}{A}_j \cdot \ddot{\underset{\sim}{C}}_j = 0 \qquad (6.37)$$

$$(\ddot{\underset{\sim}{R}}_j - \ddot{\underset{\sim}{R}}'_j) \cdot \underset{\sim}{A}_j + 2(\dot{\underset{\sim}{R}}_j - \dot{\underset{\sim}{R}}'_j) \cdot \dot{\underset{\sim}{A}}_j + (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \ddot{\underset{\sim}{A}}_j = 0 \qquad (6.38)$$

The analysis of the mechanism is carried out by using the first of the two respective equations for displacements, velocities and accelerations. This gives the relation between the input rotation and the output rotation. The translational motion at the C pair is then obtained by using the second of the two equations. This sequential solution procedure makes the expressions shorter and direct.

The displacement equations are

$$\tan(\phi_j/2) = (-b \pm \sqrt{a^2 + b^2 - c^2}\,) / (c - a) \qquad (6.39)$$

where

$$\begin{aligned} a &= \underset{\sim}{U}_{BA} \cdot \underset{\sim}{C}_j \\ b &= (\underset{\sim}{u}_B \times \underset{\sim}{A}) \cdot \underset{\sim}{C}_j \qquad (6.40) \\ c &= (\underset{\sim}{A} - \underset{\sim}{U}_{BA}) \cdot \underset{\sim}{C}_j \end{aligned}$$

and

$$S_{cj} = (\Delta\underset{\sim}{S}_j \cdot \underset{\sim}{A}_j) / (\underset{\sim}{u}_B \cdot \underset{\sim}{A}_j) \qquad (6.41)$$

where

$$\Delta\underset{\sim}{S}_j = C_1\theta_j\underset{\sim}{U}_{AR} + S\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{R}) - C_1\phi_j\underset{\sim}{U}_{BL} - S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{L}) \qquad (6.42)$$

The equations for the rotational and translational velocities are

$$\dot{\phi}_j = [-(\underset{\sim}{V}_{Aj} \cdot \underset{\sim}{A}_j) / (\underset{\sim}{V}_{Bj} \cdot \underset{\sim}{C}_j)]\, \dot{\theta}_j \qquad (6.43)$$

$$\dot{S}_{cj} = (\Delta\dot{\underset{\sim}{S}}_j \cdot \underset{\sim}{A}_j + (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \underset{\sim}{V}_{Bj}\dot{\phi}_j) / (\underset{\sim}{u}_B \cdot \underset{\sim}{A}_j) \qquad (6.44)$$

where

$$\Delta\dot{\underset{\sim}{S}}_j = \underset{\sim}{M}_{Aj}\dot{\theta}_j - \underset{\sim}{M}_{Bj}\dot{\phi}_j \qquad (6.45)$$

The acceleration equations are

$$\ddot{\phi}_j = (\underset{\sim}{W}_{Bj} \cdot \underset{\sim}{C}_j\dot{\phi}_j^{\,2} + 2\dot{\underset{\sim}{A}}_j \cdot \dot{\underset{\sim}{C}}_j + \underset{\sim}{A}_j \cdot \ddot{\underset{\sim}{C}}_j) / (-\underset{\sim}{V}_{Bj} \cdot \underset{\sim}{C}_j) \qquad (6.46)$$

$$\ddot{\underset{\sim}{S}}_{cj} = (\Delta\ddot{\underset{\sim}{S}}_j + 2(\dot{\underset{\sim}{R}}_j - \dot{\underset{\sim}{R}}'_j) \cdot \dot{\underset{\sim}{A}}_j + (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \ddot{\underset{\sim}{A}}_j) \,/\, \underset{\sim}{u}_B \cdot \underset{\sim}{A}_j \qquad (6.47)$$

where
$$\Delta\ddot{\underset{\sim}{S}}_j = \underset{\sim}{N}_{Aj}\dot{\theta}_j^2 + \underset{\sim}{M}_{Aj}\ddot{\theta}_j - \underset{\sim}{N}_{Bj}\dot{\phi}_j^2 - \underset{\sim}{M}_{Bj}\ddot{\phi}_j \qquad (6.48)$$

6.4 Analysis of the R-Sg-C Mechanism

Since the equation of motion of the R-Sg-C mechanism is in a vector form, explicit relations can be obtained by taking the dot product of the equation with the cross products of appropriate vectors. This results into scalar equations with the non-required terms made zero by the proper choice of the cross product vectors.

The equations of motion for the R-Sg-C mechanism are

$$\underset{\sim}{R}_j - \underset{\sim}{R}'_j = S_{gj}\underset{\sim}{A}_j \qquad (6.49)$$

$$\dot{\underset{\sim}{R}}_j - \dot{\underset{\sim}{R}}'_j = \dot{S}_{gj}\underset{\sim}{A}_j + S_{gj}\dot{\underset{\sim}{A}}_j \qquad (6.50)$$

and
$$\ddot{\underset{\sim}{R}}_j - \ddot{\underset{\sim}{R}}'_j = \ddot{S}_{gj}\underset{\sim}{A}_j + 2\dot{S}_{gj}\dot{\underset{\sim}{A}}_j + S_{gj}\ddot{\underset{\sim}{A}}_j \qquad (6.51)$$

The displacement equations obtained are

$$\tan(\phi_j/2) = (-b \pm \sqrt{a^2 + b^2 - c^2}\,) \,/\, (c - a) \qquad (6.52)$$

where
$$a = (\underset{\sim}{S}_j + \underset{\sim}{U}_{BL}) \cdot (\underset{\sim}{A} \times \underset{\sim}{u}_B)$$

$$b = (\underset{\sim}{S}_j + \underset{\sim}{U}_{BL}) \cdot \underset{\sim}{U}_{BA} \qquad (6.53)$$

$$c = (\underset{\sim}{u}_B \times \underset{\sim}{A}) \cdot \underset{\sim}{U}_{BL}$$

Having obtained ϕ_j, the sequential solution to obtain the other motion parameters is carried on to get

$$S_{gj} = (\Delta S_j \cdot \underset{\sim}{u}_B \times \underset{\sim}{A}) \,/\, \sin\phi_j |\underset{\sim}{u}_B \times \underset{\sim}{A}|^2 \qquad (6.54)$$

$$S_{cj} = (\Delta\underset{\sim}{S}_j \cdot \underset{\sim}{A}_j - S_{gj}) \,/\, (\underset{\sim}{u}_B \cdot \underset{\sim}{A}_j) \qquad (6.55)$$

The velocity equations are:

$$\dot{\phi}_j = \{[\underset{\sim}{M}_{Aj} \cdot \underset{\sim}{A}_j \times \underset{\sim}{u}_B] / [(S_{gj}\underset{\sim}{V}_{Bj} + \underset{\sim}{M}_{Bj}) \cdot (\underset{\sim}{A}_j \times \underset{\sim}{u}_B)]\} \dot{\theta}_j \tag{6.56}$$

$$\dot{S}_{gj} = (\Delta\dot{\underset{\sim}{S}}_j \cdot \underset{\sim}{u}_B \times \dot{\underset{\sim}{A}}_j) / (\underset{\sim}{A}_j \cdot \underset{\sim}{u}_B \times \dot{\underset{\sim}{A}}_j) \tag{6.57}$$

$$\dot{S}_{cj} = [(\Delta\dot{\underset{\sim}{S}}_j - S_{gj}\dot{\underset{\sim}{A}}_j) \cdot \underset{\sim}{A}_j - \dot{S}_{gj}] / \underset{\sim}{u}_B \cdot \underset{\sim}{A}_j \tag{6.58}$$

The acceleration equations are

$$\ddot{\phi}_j = \frac{\{[\underset{\sim}{N}_{Aj}\dot{\theta}_j^{\,2} + \underset{\sim}{M}_{Aj}\ddot{\theta}_j - (\underset{\sim}{N}_{Bj} + S_{gj}W_{Bj})\dot{\phi}_j^{\,2} - 2\dot{S}_{gj}\dot{\underset{\sim}{A}}_j] \cdot \underset{\sim}{u}_B \times \underset{\sim}{A}_j\}}{(S_{gj}\underset{\sim}{V}_{Bj} + \underset{\sim}{M}_{Bj}) \cdot (\underset{\sim}{u}_B \times \underset{\sim}{A})} \tag{6.59}$$

$$\ddot{S}_{gj} = [(\Delta\ddot{\underset{\sim}{S}}_j - 2\dot{S}_{gj}\dot{\underset{\sim}{A}}_j) \cdot (\underset{\sim}{u}_B \times \ddot{\underset{\sim}{A}}_j) / (\underset{\sim}{A}_j \cdot \underset{\sim}{u}_B \times \ddot{\underset{\sim}{A}}_j) \tag{6.60}$$

$$\ddot{S}_{cj} = [(\Delta\ddot{\underset{\sim}{S}}_j - 2\dot{S}_{gj}\dot{\underset{\sim}{A}}_j - S_{gj}\ddot{\underset{\sim}{A}}_j) \cdot \underset{\sim}{A}_j - \ddot{S}_{gj}] / \underset{\sim}{u}_B \cdot \underset{\sim}{A}_j \tag{6.61}$$

It should be noted from the preceding equations that due to the geometry and nature of the vector definitions that some terms are invariant with respect to positions. An example is

$$\underset{\sim}{u}_B \cdot \underset{\sim}{A}_j = \underset{\sim}{u}_B \cdot \underset{\sim}{A} \tag{6.62}$$

Except for the R-Sp-P mechanism, the displacement equations obtained for the other three-link function generators are quadratic in the output angle. From this, the actual angular rotations can be easily recognized from the two roots of the displacement equation. These two roots represent the two possible ways the mechanism can be assembled and the numbers themselves will reveal the correct branch of roots. With a quadratic displacement equation, positions that are not possible are shown when imaginary roots are obtained.

The analysis results, although they consist of many expressions are, however, straightforward and can be easily programmed in a computer.

6.5 Transmission Characteristics

To measure how well a mechanism would "run," some criterion or index of performance should be available. Generally, these indices of quality are based on how well the motion from one link is transmitted to another link. In most cases, only the kinematics and, in some cases, the static forces are considered in the definition of these indices. Although the dynamics of the mechanism will undoubtedly predominate the characteristics of the mechanism, some measure of the quality of the motion at the outset would be advantageous. Mechanisms that fail some pre-determined levels can readily be eliminated. These indices also allow a designer to choose among mechanisms that equally satisfy the function generation requirements.

In the following sections, the indices of transmission quality will be defined. Their applicability to the three-link function generators will be considered and the equations necessary for their application will be derived.

6.5.1 Transmission and Deviation Angles

Alt [44] defined the transmission angle for a planar four-bar mechanism as the smaller angle between the direction of the velocity difference V_{BA} of the coupler link and the direction of the absolute velocity vector V_B of the output link. This angle γ is shown in figure 6.1. It is evident from the figure that values of γ close to 90^0 will give good motion transmission.

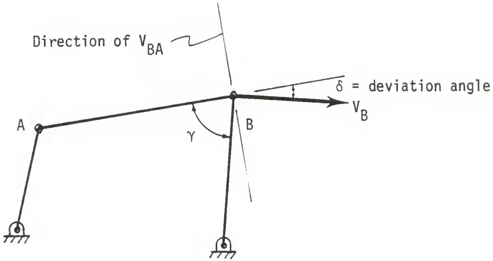


Figure 6.1 Transmission and deviation angle for a planar four-bar mechanism

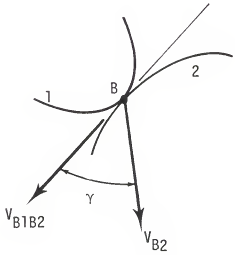


Figure 6.2 Transmission angle for a higher pair contact

Taking a different view, A. Bock [45] defined the deviation angle δ as the angle between the line of action of the static force and the direction of the velocity at the point of connection. This deviation angle δ is shown in the same planar four bar of figure 6.1. For this particular case (links of a four-bar mechanism are two-force members), $\delta + \gamma = 90^0$. Deviation angle γ close to zero indicates good motion transmission.

The concepts of transmission angle γ and deviation angle δ can be applied to the three-link function generators without modification even with the presence of the intermediate higher pairs.

For the higher pairs considered, the transmission angle γ is defined as the smaller angle between the relative velocity of the contacting elements at the point of contact and the velocity of the point of contact considered fixed to the driven link. This transmission angle is shown in figure 6.2.

The deviation angle δ, for a higher pair, can be defined as the angle between the velocity of the point of contact as fixed to the driven element and the common normal at the contacting surfaces at the point of contact. This is shown in figure 6.3. The effect of a friction angle μ to the deviation angle can also be readily incorporated. Figure 6.4 shows the actual deviation angle $\delta' = \delta + \mu$. At some position however, the velocities of both contacting elements at the point of contact would be the same so that $\delta' = \delta$. In any case, the deviation angle without the effect of the friction angle can be used for comparison purposes and also for obtaining approximate values.

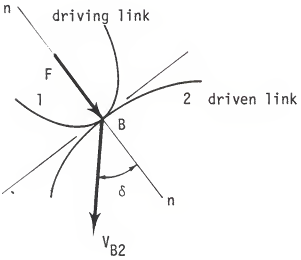


Figure 6.3 Deviation angle for a higher pair contact

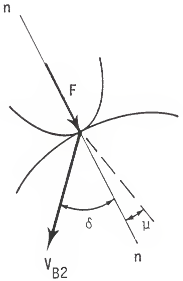


Figure 6.4 Deviation angle with friction angle

6.5.2 Applications to the Three-Link Function Generators

From section 6.5.1, it is seen that the concepts of transmission angle γ and deviation angle δ provide a way of measuring the transmission qualities of a mechanism. They are derived from purely geometric and kinematic considerations, and can readily be applied to the four three-link mechanisms with the higher pairs.

Due to the nature of the higher pair constraint equations, the deviation angle is convenient for the sphere-plane (Sp) and cylinder-plane (Cp) pairs. The transmission angle is better suited for the sphere-groove (Sg) pair because the Sg pair constraint equation is the only one that considers the relative velocity of the pair elements explicitly. The other considerations of line of action directions and points of contact will be shown in the succeeding sections.

6.6 Derivation of the Deviation Angle and Transmission Angle for the Three-Link Function Generators

To obtain the deviation angle and transmission angle in the sphere-plane (Sp), cylinder-plane (Cp) and sphere-groove (Sg) higher pairs, the following three things must be determined:

1. Point of contact.
2. Direction of the force that tends to move the other body normal to the common tangents of contacting surfaces or the relative velocity between the contacting elements at the point of contact.
3. Velocity of the point of contact considered fixed to the driven body.

The deviation angle δ or transmission angle γ will now be derived for the three-link function generators namely the R-Sp-R, R-Sp-P, R-Cp-C and the R-Sg-C. The reader is referred to figures 3.5, 3.6 and 3.7 for the detailed geometry of these mechanisms. It will be assumed that an analysis has been carried out so that displacements and velocities are already known.

6.6.1 Deviation Angle δ for the R-Sp-R and R-Sp-P Mechanisms

The three items necessary for the determination of δ are shown in figure 6.5 for a sphere-plane pair. Note that the point of contact in this pair is well defined. Also, the direction of the force applied by the sphere to the plane is in the same direction as the common normal of the two surfaces in contact.

Figure 6.6 shows the two possible locations of the actual point of contact P_j. This point P_j is considered fixed to the plane part of the pair (link 2) and is determined as

$$\underset{\sim}{P}_j = \underset{\sim}{R}_j \pm r\underset{\sim}{A}_j \tag{6.63}$$

where r is the radius of the sphere and

$\underset{\sim}{A}_j$ is normal to the plane

The plus or minus sign in equation (6.63) is determined by the direction of $\underset{\sim}{A}$ with respect to angular velocity $\dot{\phi}$ of link 2. At the initial position, the sign is determined with respect to the direction of the angular velocity. This sign is kept the same at any jth position as long as the direction of the angular velocity $\dot{\phi}_j$ does not change. Any sign change of $\dot{\phi}_j$ is then followed by a corresponding change of sign in equation (6.63).

For both the R-Sp-R and R-Sp-P mechanism, the velocity of point P_j defined by $\underset{\sim}{P}_j$ is expressed as a function of the parameters of link 2. The following expressions are derived for the velocity of P_j:

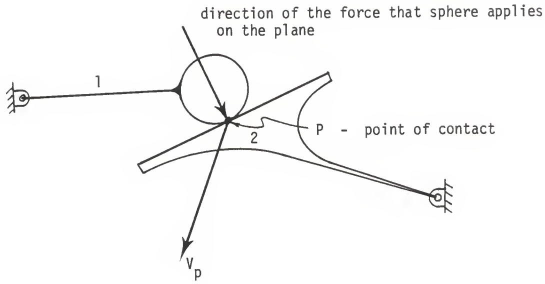


Figure 6.5 Determination of deviation angle for a sphere-plane pair

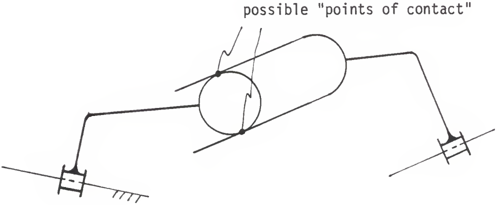


Figure 6.6 Possible points of contact in a sphere-plane pair

For the R-Sp-R mechanism

$$\dot{\underset{\sim}{P}}_j = (\underset{\sim}{u}_B \times \underset{\sim}{P}_j)\dot{\phi}_j \tag{6.64}$$

For the R-Sp-P mechanism

$$\dot{\underset{\sim}{P}}_j = \dot{S}_{cj}\underset{\sim}{u}_B \tag{6.65}$$

As mentioned earlier, the common normal to the sphere and plane surfaces defines the direction of the force of the sphere on the plane. This direction is defined by $\underset{\sim}{A}_j$ for the R-Sp-R mechanism, and by $\underset{\sim}{A}$ for the R-Sp-P mechanism. Therefore, the deviation angle δ_j for the R-Sp-R mechanism is

$$\cos\delta_j = \underset{\sim}{A}_j \cdot (\underset{\sim}{u}_B \times \underset{\sim}{P}_j) \;/\; |\underset{\sim}{u}_B \times \underset{\sim}{P}_j| \tag{6.66}$$

And for the R-Sp-P mechanism, the deviation angle δ is

$$\cos\delta = \underset{\sim}{A} \cdot \underset{\sim}{u}_B \tag{6.67}$$

6.6.2 Deviation Angle for the R-Cp-C Mechanism

Unlike the sphere-plane (Sp) pair, the cylinder-plane (Cp) has a line contact between its pair elements. This makes the point of contact difficult to pinpoint exactly. However, an approximate location would be a point that is midway along the line of contact. Similar to a revolute pair in a planar four-bar wherein the line of action passes through the revolute axis, the line of action of the force in the Cp pair can be considered to pass through the axis of the cylinder. Thus, the point of contact is located as shown in figure 6.7. The location of this point must, however, be determined. Figure 6.8 shows the initial and jth position of the pair elements in the Cp pair. By defining a

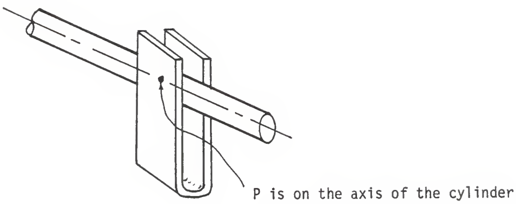


Figure 6.7 Point of contact in a cylinder-plane pair

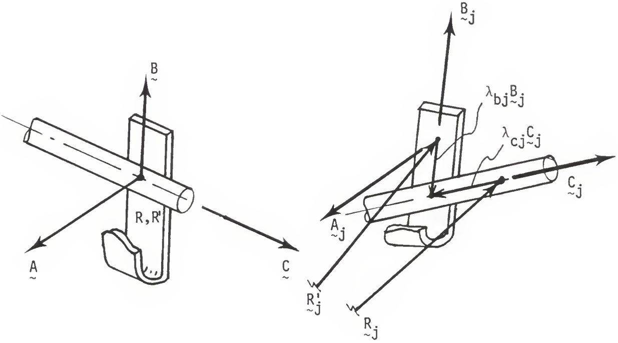


Figure 6.8 Initial and (jth) positions of the Cp pair elements

unit vector $\underset{\sim}{B}$ that is parallel to the plane and also perpendicular to $\underset{\sim}{C}$ (i.e., $\underset{\sim}{B} = \underset{\sim}{A} \times \underset{\sim}{C}$), the vector $\underset{\sim}{P}_j$ which locates the point P_j can be obtained as follows:

From figure 6.7,

$$\underset{\sim}{P}_j = \underset{\sim}{R}_j + \lambda_{cj}\underset{\sim}{C}_j \qquad (6.68)$$

and

$$\underset{\sim}{P}_j = \underset{\sim}{R}'_j + \lambda_{bj}\underset{\sim}{B}_j \qquad (6.69)$$

Subtracting equation (6.69) from (6.68) results in

$$(\underset{\sim}{R}_j - \underset{\sim}{R}'_j) + \lambda_{cj}\underset{\sim}{C}_j = \lambda_{bj}\underset{\sim}{B}_j \qquad (6.70)$$

Taking the cross product of equation (6.70) with $(\underset{\sim}{A}_j \times \underset{\sim}{B}_j)$ and recognizing that $(\underset{\sim}{A}_j \times \underset{\sim}{B}_j)$ is perpendicular to $\underset{\sim}{B}_j$ will result in

$$(\underset{\sim}{A}_j \times \underset{\sim}{B}_j) \cdot (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) + \lambda_{cj}(\underset{\sim}{A}_j \times \underset{\sim}{B}_j) \cdot \underset{\sim}{C}_j = 0 \qquad (6.71)$$

or

$$\lambda_{cj} = -(\underset{\sim}{A}_j \times \underset{\sim}{B}_j) \cdot (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \,/\, (\underset{\sim}{A}_j \times \underset{\sim}{B}_j) \cdot \underset{\sim}{C}_j \qquad (6.72)$$

Using similar vector manipulations again to equation (6.70), λ_{bj} is obtained as

$$\lambda_{bj} = (\underset{\sim}{A}_j \times \underset{\sim}{C}_j) \cdot (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \,/\, (\underset{\sim}{A}_j \times \underset{\sim}{C}_j) \cdot \underset{\sim}{B}_j \qquad (6.73)$$

$\underset{\sim}{P}_j$ is then solved by substituting either λ_{cj} into equation (6.68) or λ_{bj} into equation (6.69).

The velocity of point P_j as defined by parameters of link 2 is obtained as

$$\dot{\underset{\sim}{P}}_j = (\underset{\sim}{u}_B \times \underset{\sim}{P}_j)\dot{\phi}_j + \underset{\sim}{u}_B\dot{S}_{cj} \qquad (6.74)$$

The common normal to the plane and cylinder which also defines the line of action of the force that the cylinder applies to the plane is vector $\underset{\sim}{A}_j$ so that the deviation angle δ_j is

$$\cos\delta_j \;=\; \underset{\sim}{A}_j \cdot [(\underset{\sim}{u}_B \times \underset{\sim}{P}_j)\dot{\phi}_j + \underset{\sim}{u}_B\dot{S}_{cj}] \;/\; |(\underset{\sim}{u}_B \times \underset{\sim}{P}_j)\dot{\phi}_j + \underset{\sim}{u}_B\dot{S}_{cj}| \qquad (6.75)$$

The deviation angle δ_j obtained for the R-Cp-C mechanism is dependent on the actual location of the points of contact. Recall that the R-Cp-C mechanism can have the plane located anywhere along the cylinder as long as the orientation of the plane is preserved. This suggests that the actual locations of the points of contact are dependent on where the plane was located initially. Because different points on link 2 (the plane part of the pair) will have different velocities, the deviation angle will also be dependent on the initial location of the plane. Thus, the "same" R-Cp-C mechanism can have different deviation angles.

The situation discussed above presents a way of obtaining acceptable deviation angles. The deviation angles for a mechanism, whose location at the plane is specified arbitrarily or set by the vector $\underset{\sim}{R}$, are determined and if they are not acceptable, then the plane can be relocated to give better transmission angles.

6.6.3 Transmission Angle for the R-Sg-C Mechanism

In the case of the sphere-groove pair, the center of the sphere can also be assumed as the "point of contact." The line of action of the force that the sphere exerts on the groove will always be perpendicular to the axis of the groove. The actual deviation in a plane perpendicular to the groove is, however, difficult to determine. This situation is resolved by using the transmission angle instead. The "point of contact" defined by $\underset{\sim}{P}_j$ is really $\underset{\sim}{R}_j$. The velocity must then be determined by the parameters of link 2. Therefore, the velocity of point P_j is

$$\underset{\sim}{P}_j = (\underset{\sim}{u}_B \times \underset{\sim}{R}_j)\dot{\phi}_j + \underset{\sim}{u}_B\dot{S}_{cj} \tag{6.76}$$

The direction of the relative velocity vector is in $\underset{\sim}{A}_j$ so that the transmission angle γ_j is obtained as

$$\cos\gamma_j = \underset{\sim}{A}_j \cdot [(\underset{\sim}{u}_B \times \underset{\sim}{R}_j)\dot{\phi}_j + \underset{\sim}{u}_B\dot{S}_{cj}] \;/\; |(\underset{\sim}{u}_B \times \underset{\sim}{R}_j)\dot{\phi}_j + \underset{\sim}{u}_B\dot{S}_{cj}| \tag{6.77}$$

It should be mentioned here that the deviation angles and the transmission angles for the mechanisms have been derived with the assumption that link 1 is the driving or input link. These deviation and transmission angles are not the same if link 1 is the driven link. To obtain these other set of deviation and transmission angles, the expression for $\underset{\sim}{P}_j$ will change so that it is now the velocity of the same point P_j fixed in link 1.

CHAPTER 7

DESIGN OF THE PAIR ELEMENTS

The objective in this chapter is to determine the pertinent dimensions and shape of the higher pair elements to insure that the motion of the mechanism as constrained by the higher pair is preserved. This involves preventing the pair elements from interfering with each other (other than the required contact). At the same time also, the higher pairs must not be allowed to separate.

The derivations that will follow are based on geometry and motion (displacement) considerations only. No attempt would be made to include static and dynamic load concerns. It will also be assumed that a displacement analysis (see Chapter 6) has been carried out for each of the mechanisms.

7.1 Design of the Sphere-Plane Pair Elements

The design of the sphere-plane (Sp) pair elements involve determining the extent or contour of the plane, the sphere radius and the orientation of the rod connecting the sphere and the revolute pair of link 1. These requirements are shown in figure 7.1.

To be able to fully define displacements and rotations within the pair, two unit vectors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are fixed to the plane such that $\underset{\sim}{A}$, $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are mutually perpendicular to each other. These vectors together with $\underset{\sim}{A}$ are shown at their initial positions in figure 7.2. The vectors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are derived from the following equations:

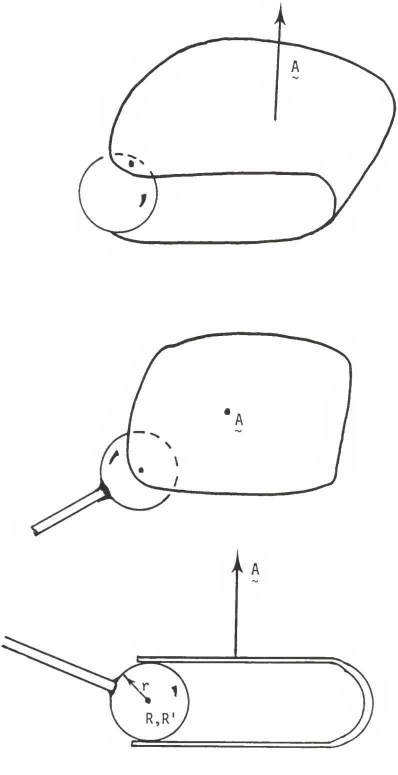


Figure 7.1 Requirements for the design of the sphere-plane pair elements.

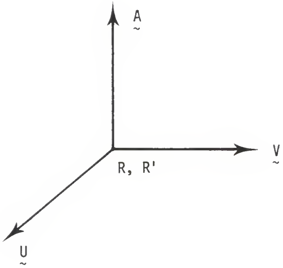


Figure 7.2 The in-plane coordinate system of the Sp pair

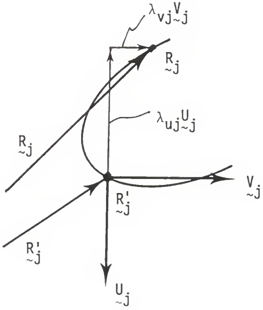


Figure 7.3 Determination of the plane contour

$$\underset{\sim}{A} \cdot \underset{\sim}{U} = 1 + u_1 d_1 + u_2 d_2 = 0 \tag{7.1}$$

and

$$\underset{\sim}{A} \times \underset{\sim}{U} = \underset{\sim}{V} \tag{7.2}$$

where

$$\left.\begin{aligned} \underset{\sim}{U} &= u_x \hat{i} + u_y \hat{j} + u_z \hat{k} \\ \underset{\sim}{V} &= v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \end{aligned}\right. \tag{7.3}$$

$$\left.\begin{aligned} {u_x}^2 + {u_y}^2 + {u_z}^2 &= 1 \\ {v_x}^2 + {v_y}^2 + {v_z}^2 &= 1 \end{aligned}\right. \tag{7.4}$$

$$\left.\begin{aligned} u_1 &= u_y/u_x \, ; \quad u_2 = u_z/u_x \\ v_1 &= v_y/v_x \, ; \quad v_2 = v_z/v_x \end{aligned}\right. \tag{7.5}$$

and where d_1 and d_2 are constants given by equation (4.21)

Vector $\underset{\sim}{U}$ is obtained by arbitrarily specifying either u_1 or u_2 and then solving for the other parameter using equation (7.1). With $\underset{\sim}{U}$ known, equation (7.2) is used to determine the vector $\underset{\sim}{V}$.

These three vectors with their origin at R' can be considered as the in-plane coordinate system R'AUV.

7.1.1 The Contour of the Plane in the Sp Pair

The contour of the plane is obtained by determining the path of the point R (the sphere center) at any jth position with respect to the in-plane coordinate system. Using the results of a displacement analysis, the vectors $\underset{\sim}{R}_j$ and $\underset{\sim}{R}'_j$ are used to locate the sphere center R and the initially coincident point R' of the plane respectively. Thus, the vector $(\underset{\sim}{R}_j - \underset{\sim}{R}'_j) = \Delta\underset{\sim}{R}_j$ gives the location of the sphere center R with respect to the origin of the in-plane coordinate system R'AUV. To fully define the location of this point, R_j (sphere center), the components of

$\underset{\sim}{\Delta R}_j$ in the $\underset{\sim}{U}_j$ and $\underset{\sim}{V}_j$ axes must be determined. This task is achieved by finding two scalars λ_{uj} and λ_{vj} that multiply $\underset{\sim}{U}_j$ and $\underset{\sim}{V}_j$ to give the respective components of $\underset{\sim}{\Delta R}_j$. The method is illustrated in figure 7.3.

The expressions for $\underset{\sim}{U}_j$ and $\underset{\sim}{V}_j$ are obtained from the vector rotations of $\underset{\sim}{U}$ and $\underset{\sim}{V}$ about $\underset{\sim}{u}_B$ by the angle ϕ_j. The results are

$$\underset{\sim}{U}_j = \underset{\sim}{U} + C_1\phi_j \underset{\sim}{U}_{BU} + S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{U}) \tag{7.6}$$

and
$$\underset{\sim}{V}_j = \underset{\sim}{V} + C_1\phi_j \underset{\sim}{U}_{BV} + S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{V}) \tag{7.7}$$

where
$$\underset{\sim}{U}_{BU} = (\underset{\sim}{u}_B \times \underset{\sim}{U}) \times \underset{\sim}{u}_B \tag{7.8}$$

$$\underset{\sim}{U}_{BV} = (\underset{\sim}{u}_B \times \underset{\sim}{V}) \times \underset{\sim}{u}_B \tag{7.9}$$

and, $C_1\phi_j$ and $S\phi_j$ are defined in equation (4.19)

From figure 7.3, the following equation is obtained:

$$\underset{\sim}{R}_j = \underset{\sim}{R}'_j + \lambda_{uj}\underset{\sim}{U}_j + \lambda_{vj}\underset{\sim}{V}_j \tag{7.10}$$

The scalars λ_{uj} and λ_{vj} are obtained by taking the dot product of equation (7.10) with $\underset{\sim}{U}_j$ and $\underset{\sim}{V}_j$ respectively. Recognizing that $\underset{\sim}{V}_j$ and $\underset{\sim}{U}_j$ are unit vectors and are perpendicular to each other, λ_{uj} and λ_{vj} are

$$\lambda_{uj} = (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \underset{\sim}{U}_j \tag{7.11}$$

and

$$\lambda_{vj} = (\underset{\sim}{R}_j - \underset{\sim}{R}'_j) \cdot \underset{\sim}{V}_j \tag{7.12}$$

Enough (jth) positions are then evaluated to define the outline of the plane as shown in figure 7.4. From the resulting outline of the plane, a general idea of the orientation of the sphere-rod -- the connection of link 1 to the sphere -- can be made. Logically, the sphere-rod must

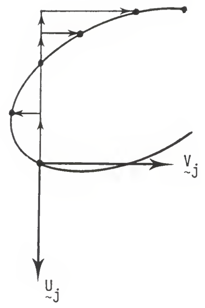


Figure 7.4 The R_j-s points defining the plane contour

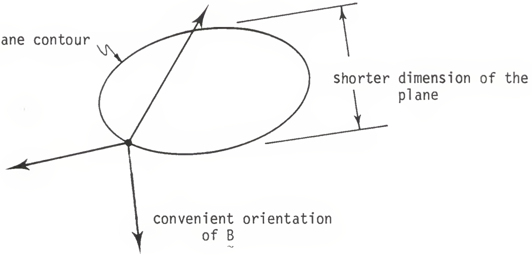

plane contour
shorter dimension of the plane
convenient orientation of $\underset{\sim}{B}$

Figure 7.5 Initial orientation of the sphere-rod

be along the "width" of the plane to minimize the possibility of link interference. The sphere-rod must also be located so that, at the initial position, the sphere-rod is not within the plane. These two conditions are shown in figure 7.5.

7.1.2 The Sphere-Rod and the Sphere Radius

The objective in this section is to determine the orientation of the sphere-rod and to determine the minimum radius of the sphere.

With the purpose of making the solution procedure as simple as possible, the orientation of the sphere-rod is initially assumed and the radius of the sphere is then determined. Changes in the assumed orientation of the sphere-rod are later made based on the results obtained.

As mentioned in the previous section, the sphere-rod is oriented so that is is along the shorter dimension of the plane outline and is also initially made to lie in the plane of the pair.

The unit vector $\underset{\sim}{B}$ considered fixed to link 1 is now introduced to define the assumed orientation of the sphere-rod. The sense of the vector $\underset{\sim}{B}$ is also set to be directed away from the plane.

Figure 7.6 and 7.7 show the initial and jth positions of the vectors $\underset{\sim}{B}$, $\underset{\sim}{A}$, $\underset{\sim}{U}$ and the location of points R and R'. Figure 7.7 also shows the intersection of the line of action of the component in the plane of $\underset{\sim}{B}_j$ with the outline of the plane. By virtue of the motion of link 1 (rotation θ_j about axis $\underset{\sim}{u}_A$), the vector $\underset{\sim}{B}_j$ is given by

$$\underset{\sim}{B}_j = \underset{\sim}{B} + C_1\theta_j\underset{\sim}{U}_{AB} + S\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{B}) \qquad (7.13)$$

where $$\underset{\sim}{U}_{AB} = (\underset{\sim}{u}_A \times \underset{\sim}{B}) \times \underset{\sim}{u}_A \qquad (7.14)$$

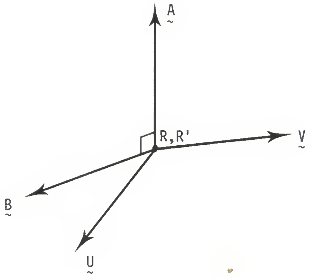


Figure 7.6 Vectors $\underset{\sim}{A}$, $\underset{\sim}{U}$, $\underset{\sim}{V}$ and $\underset{\sim}{B}$ at the initial position

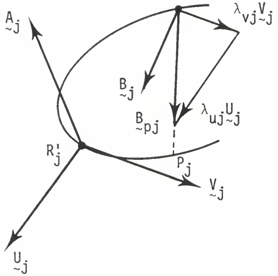


Figure 7.7 (Jth) position of $\underset{\sim}{B}_j$ with respect to the in-plane coordinate system.

The equations for $\underset{\sim}{U}_j$ and $\underset{\sim}{V}_j$ are given by equations (7.6) and (7.7) while that for $\underset{\sim}{A}_j$ is recalled from previous derivations as

$$\underset{\sim}{A}_j = \underset{\sim}{A} + C_1\phi_j\underset{\sim}{U}_{BA} + S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{A}) \tag{7.15}$$

The vector $\underset{\sim}{B}_j$ is now resolved into its components in the directions of $\underset{\sim}{A}_j$, $\underset{\sim}{U}_j$ and $\underset{\sim}{V}_j$.

$$\underset{\sim}{B}_j = \lambda'_{aj}\underset{\sim}{A}_j + \lambda'_{uj}\underset{\sim}{U}_j + \lambda'_{vj}\underset{\sim}{V}_j \tag{7.16}$$

The scalars λ'_{aj}, λ'_{uj} and λ'_{vj} are obtained by taking the dot product of equation (7.16) with $\underset{\sim}{A}_j$, $\underset{\sim}{U}_j$ and $\underset{\sim}{V}_j$ respectively. Thus,

$$\lambda'_{aj} = \underset{\sim}{B}_j \cdot \underset{\sim}{A}_j \tag{7.17}$$

$$\lambda'_{uj} = \underset{\sim}{B}_j \cdot \underset{\sim}{U}_j \tag{7.18}$$

$$\lambda'_{vj} = \underset{\sim}{B}_j \cdot \underset{\sim}{V}_j \tag{7.19}$$

Having obtained λ'_{uj} and λ'_{vj}, the component of $\underset{\sim}{B}_j$ along the plane called $\underset{\sim}{B}_{pj}$ is determined as

$$\underset{\sim}{B}_{pj} = \lambda'_{uj}\underset{\sim}{U}_j + \lambda'_{vj}\underset{\sim}{V}_j \tag{7.20}$$

Since the determination of the plane outline requires graphically drawing the plane by connecting the points contained in the outline, a graphical procedure would not be out of line at this point. Using a graphical method will also avoid cumbersome derivations and complex equations.

The line of action of vector $\underset{\sim}{B}_{pj}$ will intersect the outline of the plane at the points $\underset{\sim}{R}_j$ and $\underset{\sim}{P}_j$. The point P_j is determined graphically by going from point R_j in the direction of $\underset{\sim}{B}_{pj}$ until it intersects the plane outline. Note that point P_j may not exist at all, as depicted in figure 7.8.

The ratio of the length from R_j to P_j to the length of $\underset{\sim}{B}_{pj}$ is

$$\lambda_{1j} = (\overline{R_jP_j}) \, / \, |\underset{\sim}{B}_{pj}| \qquad (7.21)$$

$$\text{or} \qquad \lambda_{1j} = (\overline{R_jP_j}) \, / \sqrt{\lambda_{uj}'^{\,2} + \lambda_{vj}'^{\,2}} \qquad (7.22)$$

Having obtained λ_{1j}, the vector connecting the point P_j to the line of action of $\underset{\sim}{B}_j$ is now determined as $\lambda_{1j}\lambda_{aj}'A_j$. Figure 7.9 is a view of the (jth) position of the higher pair looking perpendicular to the plane formed by $\underset{\sim}{B}_j$ and $\underset{\sim}{A}_j$. The minimum radius of the sphere that will make the rod clear the plane at this (jth) position is

$$r_j = | \lambda_{1j}\lambda_{aj}' | \qquad (7.23)$$

$$\text{or} \qquad r_j = \lambda_{aj}'(\overline{R_jP_j}) \, / \sqrt{\lambda_{uj}^{\,2} + \lambda_{vj}'^{\,2}} \qquad (7.24)$$

This is for zero rod thickness. For realistic design, the radius of the actual rod thicknesss times the secant of the angle between the rod and the plane should be considered. This is shown in figure 7.10.

Equation (7.24) is thus modified to include the effect of the sphere-rod radius r_r.

$$r_j' = \lambda_{aj}'(R_jP_j) \, / \sqrt{\lambda_{uj}'^{\,2} + \lambda_{vj}'^{\,2}} + r_r \sec\theta_{aj} \qquad (7.25)$$

The sphere radius for different (jth) positions are then determined - from which the maximum r_j' is chosen. This value of r_j' is the minimum radius the sphere can have so that interference is avoided at all positions. Note that a point is obtained as the sphere radius if the point P_j does not exist.

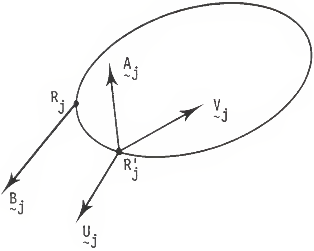


Figure 7.8 Non-existence of point P_j

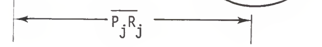

$\lambda_1\lambda'_{aj}\underset{\sim}{A}_j$
$\underset{\sim}{B}_j$
θ_{aj}
P_j
R_j
$\underset{\sim}{B}_{pj}$
$\overline{P_jR_j}$

Figure 7.9 Determination of sphere radius in the Sp pair

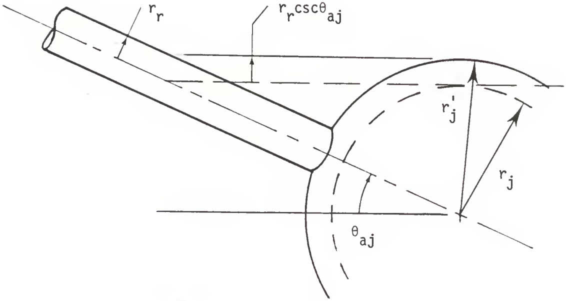


Figure 7.10 Effect of sphere-rod radius r_r to the sphere radius r_j

The maximum value of λ_{1j} will also determine the necessary length of the sphere-rod before any bends can be made.

The different values of the angle θ_{aj} (considered positive if λ'_{aj} is positive) shown in figure 7.9 are now used to make adjustments in the assumed sphere-rod orientation. Adjustments would be necessary if the required minimum radius of the sphere is not acceptable to the designer. For example, if all of the θ_{aj}-s are positive, then the vector $\underset{\sim}{B}$ can be rotated by half of $(-\theta_{aj})_{max}$ along the plane defined by $\underset{\sim}{B}$ and $\underset{\sim}{A}$. This would then halve the required minimum radius of the sphere.

7.2 Design of the Cylinder-Plane Pair Elements

As mentioned previously, the location of the pair in the R-Cp-C mechanism is not unique. The plane can be located anywhere along the axis of the cylinder as long as its orientation is preserved.

The objective of designing the pair elements in the cylinder-plane pair is to make sure that the cylinder and the plane will not separate. The procedure of designing the pair elements was essentially carried out in the determination of the deviation angle for the R-Cp-C mechanism (see section 6.6.2).

The method will be presented briefly to include other options available to the designer. Figure 7.11 shows the unit vectors $\underset{\sim}{A}$, $\underset{\sim}{C}$, $\underset{\sim}{B}$ and the chosen location of the points R and R'. The procedure in section 6.6.2 assumed that the vector $\underset{\sim}{B}$ was perpendicular to both $\underset{\sim}{A}$ and $\underset{\sim}{C}$ for convenience. However, the only requirement that the unit vector $\underset{\sim}{B}$ must satisfy is that it be perpendicular to vector $\underset{\sim}{A}$. The vector $\underset{\sim}{B}$ can be determined with ease by specifying the angle β it makes with $\underset{\sim}{C}$ as measured about vector $\underset{\sim}{A}$. Considering vector $\underset{\sim}{C}$ to be rotated about $\underset{\sim}{A}$ by the angle β,

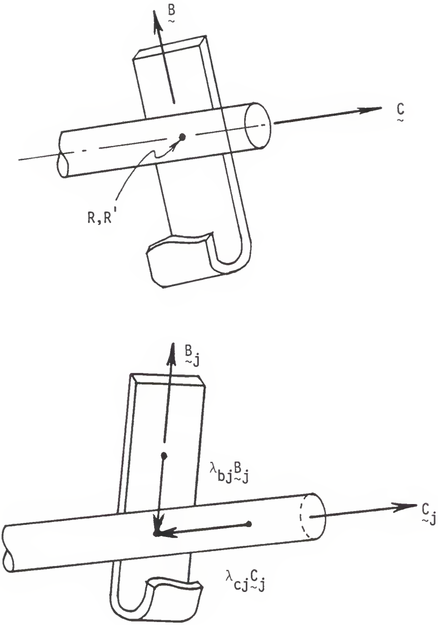


Figure 7.11 Initial and (jth) positions of the elements in a cylinder-plane pair

$$\underset{\sim}{B} = \underset{\sim}{C} + C_1\beta(\underset{\sim}{A} \times \underset{\sim}{C}) \times \underset{\sim}{A} + S\beta(\underset{\sim}{A} \times \underset{\sim}{C}) \qquad (7.26)$$

where $C_1\beta = \cos\beta - 1$

$S\beta = \sin\beta$

Equation (7.26) is simplified as

$$\underset{\sim}{B} = \cos\beta\underset{\sim}{C} + \sin\beta(\underset{\sim}{A} \times \underset{\sim}{C}) \qquad (7.27)$$

The task now is to determine the length of the cylinder and also to determine the extent of the plane in the direction of vector $\underset{\sim}{B}$. The procedure is similar to that shown in section 6.6.2 but the resulting equations are slightly different because vector $\underset{\sim}{B}$ is not necessarily perpendicular to $\underset{\sim}{C}$. Figure 7.11 of the (jth) position of the (Cp) pair shows the two scalars λ_{cj} and λ_{bj}. These two scalars define the required extent of the plane and length of the cylinder to prevent separation at this (jth) position. From the figure,

$$\underset{\sim}{R}_j - \underset{\sim}{R}'_j + \lambda_{cj}\underset{\sim}{C}_j = \lambda_{bj}\underset{\sim}{B}_j \qquad (7.28)$$

where $$\underset{\sim}{C}_j = \underset{\sim}{C} + C_1\theta_j\underset{\sim}{U}_{AC} + S\theta_j(\underset{\sim}{u}_A \times \underset{\sim}{C}) \qquad (7.29)$$

$$\underset{\sim}{B}_j = \underset{\sim}{B} + C_1\phi_j\underset{\sim}{U}_{BB} + S\phi_j(\underset{\sim}{u}_B \times \underset{\sim}{B}) \qquad (7.30)$$

λ_{cj} and λ_{bj} are obtained by taking the dot product of equation (7.28) with $(\underset{\sim}{B}_j \times \underset{\sim}{R}_j)$ and $(\underset{\sim}{C}_j \times \underset{\sim}{R}'_j)$ respectively. This results in

$$\lambda_{cj} = \underset{\sim}{R}'_j \cdot (\underset{\sim}{B}_j \times \underset{\sim}{R}_j) \,/\, \underset{\sim}{C}_j \cdot (\underset{\sim}{B}_j \times \underset{\sim}{R}_j) \qquad (7.31)$$

and

$$\lambda_{bj} = \underset{\sim}{R}_j \cdot (\underset{\sim}{C}_j \times \underset{\sim}{R}'_j) \,/\, \underset{\sim}{B}_j \cdot (\underset{\sim}{C}_j \times \underset{\sim}{R}'_j) \qquad (7.32)$$

The scalars λ_{cj} and λ_{bj} are then evaluated for different (jth) positions. From the different values of λ_{cj} and λ_{bj}, the largest positive λ_{cj} and λ_{bj} will determine the length of the cylinder and extent of the plane in the same sense as $\underset{\sim}{C}$ and $\underset{\sim}{B}$ respectively from the points R and R'. Likewise, the negative λ_{cj} and λ_{bj} with the largest magnitudes will define the length of the cylinder and extent of the plane opposite the directions of $\underset{\sim}{C}$ and $\underset{\sim}{B}$ from the initial points R, R'.

It should be noted that the diameter of the cylinder and the width of the plane will not affect the motion of the mechanism nor will it be a concern for pair interference.

7.3 Design of the Sphere-Groove Pair Elements

The key items necessary in the design of the sphere-groove (Sg) pair for the R-Sg-C mechanism are the length of the groove and the path of the sphere-rod on the surface of the cylinder surface. The diameter of the sphere-rod can be assumed to be any convenient value.

Figure 7.12 shows a sphere-groove pair of the required length and the necessary cut-out in the cylinder surface to clear the sphere-rod.

The length of the cylinder is determined by the displacements of point R, the sphere center, with respect to the initially coincident point R'. This is given by the scalar S_{gj} which was previously determined in the displacement analysis for the R-Sg-C mechanism (see section 6.4). From the different values of S_{gj}, the positive and negative S_{gj}-s with the largest magnitudes are picked out to determine the length of the groove. Thus, the length is the sum of the magnitudes of the largest positive and negative S_{gj}-s. The length in the direction of $\underset{\sim}{A}$, measured from point R' is given by the positive S_{gj}, while the length in the direction opposite to $\underset{\sim}{A}$ is given by the negative S_{gj}. This is also shown in figure 7.12.

To determine the necessary cut-out in the surface of the cylinder, a coordinate axis R'AUV is fixed to the groove or cylinder of link 2. This coordinate axis is defined by vectors $\underset{\sim}{A}$, $\underset{\sim}{U}$ and $\underset{\sim}{V}$. The vectors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are obtained in the same way as that for the sphere-plane pair. The vector $\underset{\sim}{B}$, fixed to link 1 is now specified arbitrarily to define the orientation of the sphere-rod at the initial position. For convenience, the vector $\underset{\sim}{B}$ is chosen so that it is perpendicular to the vector $\underset{\sim}{A}$. To further simplify the derivation, the vector $\underset{\sim}{B}$ can be made equal to $\underset{\sim}{V}$ or from a different viewpoint, the vectors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are reoriented so that $\underset{\sim}{V} = \underset{\sim}{B}$ at the initial position of the mechanism. At this point, the radius of the sphere is assumed to be any convenient value r_s.

Figure 7.13 shows the (jth) position of the vectors $\underset{\sim}{A}$, $\underset{\sim}{U}$, $\underset{\sim}{V}$ and $\underset{\sim}{B}$. These (jth) vectors $\underset{\sim}{A}_j$, $\underset{\sim}{U}_j$, $\underset{\sim}{V}_j$ and $\underset{\sim}{B}_j$ are obtained from equations (7.15), (7.6), (7.7) and (7.13) respectively of section 7.1.1. The vectors $\underset{\sim}{B}_j$ is now resolved into its components in the $\underset{\sim}{A}_j$, $\underset{\sim}{U}_j$ and $\underset{\sim}{V}_j$ directions. This is done to determine the relative rotations of vector $\underset{\sim}{B}_j$ with respect to link 2 about axes $\underset{\sim}{A}_j$ and $\underset{\sim}{U}_j$. Thus,

$$\underset{\sim}{B}_j = \lambda_{aj}\underset{\sim}{A}_j + \lambda_{uj}\underset{\sim}{U}_j + \lambda_{vj}\underset{\sim}{V}_j \qquad (7.33)$$

Equations (7.17), (7.18) and (7.19) give the solutions for λ_{aj}, λ_{uj} and λ_{vj} as follows,

$$\lambda_{aj} = \underset{\sim}{B}_j \cdot \underset{\sim}{A}_j \qquad (7.17)$$

$$\lambda_{uj} = \underset{\sim}{B}_j \cdot \underset{\sim}{U}_j \qquad (7.18)$$

$$\lambda_{vj} = \underset{\sim}{B}_j \cdot \underset{\sim}{V}_j \qquad (7.19)$$

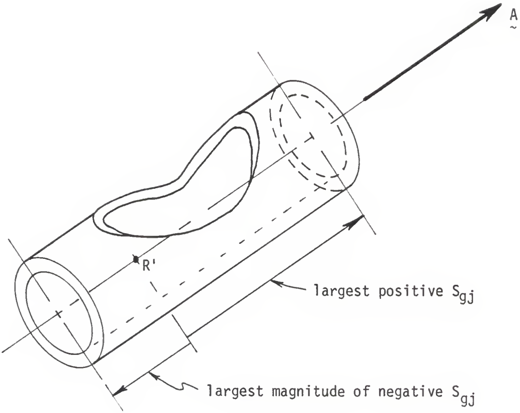

$\underset{\sim}{A}$
R'
largest positive S_{gj}
largest magnitude of negative S_{gj}

Figure 7.12 The groove part of the Sg pair

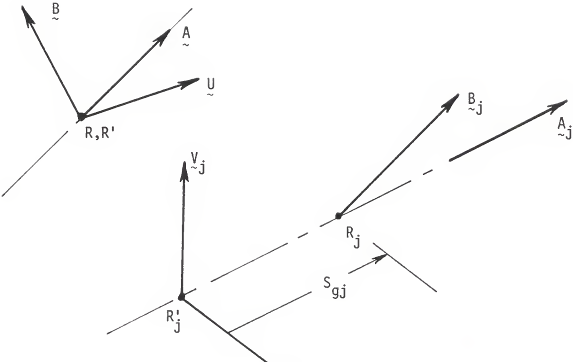


Figure 7.13 Initial and (jth) positions of vectors $\underset{\sim}{A}$, $\underset{\sim}{U}$, $\underset{\sim}{V}$ and $\underset{\sim}{B}$.

The components of $\underset{\sim}{B}_j$ in the $\underset{\sim}{A}_j$, $\underset{\sim}{U}_j$ and $\underset{\sim}{V}_j$ directions, together with the rotations β_{aj} about the $\underset{\sim}{A}_j$ axis and β_{abj} about the $(\underset{\sim}{A}_j \times \underset{\sim}{B}_j)$ axis are shown in figure 7.14. The view of the (jth) position, looking towards $\underset{\sim}{A}_j$ and looking towards the plane containing $\underset{\sim}{A}_j$ and $\underset{\sim}{B}_j$, are shown in figures 7.15 and 7.16, respectively. From these two figures, the necessary cut-out on the cylinder surface is shown as the intersection of the line of action of vector $\underset{\sim}{B}_j$ with the surface of the cylinder of radius r_s. The point of intersection can be defined by the angle β_{aj} which is the rotation of $\underset{\sim}{B}_j$ about the axis of the cylinder. These parameters β_{aj} and S_{Gj} are obtained with respect to the groove coordinate system R'AUV as

$$\beta_{aj} = \tan^{-1} (\lambda_{vj} / \lambda_{uj}) \tag{7.34}$$

and

$$S_{Gj} = S_{gj} + r_s \lambda_{aj} / \sqrt{\lambda_{uj}^2 + \lambda_{vj}^2} \tag{7.35}$$

To include the effect of the actual sphere-rod radius r_r and the cylinder wall thickness t, the value of S_{Gj} is modified as

$$S'_{Gj} = S_{gj} + (r_s + t)\lambda_{aj} / \sqrt{\lambda_{uj}^2 + \lambda_{vj}^2} + r_r \sqrt{\lambda_{uj}^2 + \lambda_{vj}^2} / \lambda_{aj} \tag{7.36}$$

Several (jth) positions are then evaluated to obtain the profile of the necessary cut-out on the cylinder surface.

It should be noted that any value of the sphere radius r_s could have been chosen and the results would be the same except for a scale change in the second expression in equation (7.35). It is also possible that the sphere-rod would clear the end of the groove and have no intersection with the cylinder surface. This happens whenever $|S_{Gj}| > |S_{gj}|_{\text{maximum}}$.

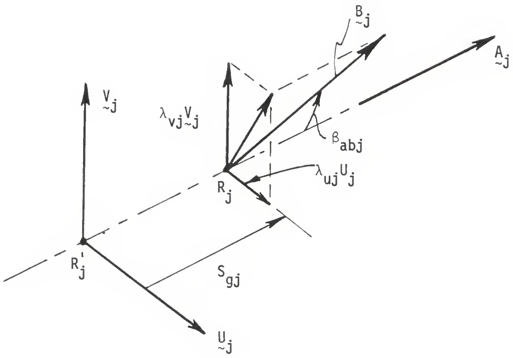


Figure 7.14 Components of $\underset{\sim}{B}_j$ in the in-groove coordinate system

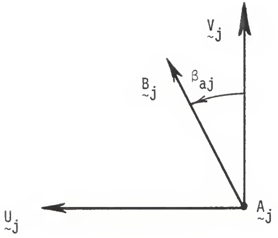


Figure 7.15 View of the (jth) position of $\underset{\sim}{B}_j$ looking at the vector $\underset{\sim}{A}$

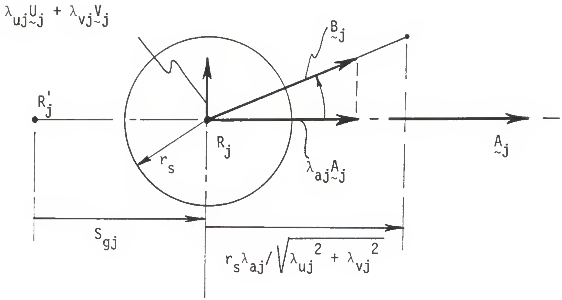

$\lambda_{uj}\underset{\sim}{U}_j + \lambda_{vj}\underset{\sim}{V}_j$
$\underset{\sim}{B}_j$
R_j'
R_j
r_s
$\lambda_{aj}\underset{\sim}{A}_j$
$\underset{\sim}{A}_j$
S_{gj}
$r_s\lambda_{aj}/\sqrt{\lambda_{uj}^2 + \lambda_{vj}^2}$

Figure 7.16 The intersection of the sphere-rod with the surface of the cylinder

CHAPTER 8

NUMERICAL EXAMPLES

In this chapter, the synthesis of the four function generators for their maximum number of positions will be shown to illustrate the procedures that were developed.

8.1 Five-MSP Synthesis of a Revolute--Sphere-Plane--Revolute Mechanism

The mechanism to be synthesized is intended to satisfy the following prescribed five-MSP specifications (P-P-P-PV):

$$\begin{array}{llll} \text{At} & j = 1, & \theta_1 = 0 & \phi_1 = 0 \\ & j = 2 & \theta_2 = 30° & \phi_2 = 15° \\ & j = 3, & \theta_3 = 90° & \phi_3 = 50° \\ & j = 4, & \theta_4 = 150° & \phi_4 = 70° \\ & j = 5, & \theta_5 = 150° & \phi_5 = 70° \\ & & \dot{\theta}_5 = 1 & \dot{\phi}_5 = .2 \end{array} \qquad (8.1)$$

The unknowns for this synthesis case R5 are r_1, r_2. r_3 and d_1 (see figure 3.2, equations (4.5) and (4.6), and section 5.5.3).

Letting $d_2 = 0$

and $$\underset{\sim}{u}_B = .3162\hat{i} + 0\hat{j} + .9487\hat{k}, \qquad (8.2)$$

the K_{ij}^{n} terms are obtained from the specifications of θ_j, ϕ_j, $\dot{\theta}_j$, $\dot{\phi}_j$ and $\underset{\sim}{u}_B$ to get the following system of equations:

$$\begin{aligned}
&(.3067 + 2.4554d_1)r_1 + (-2.1776 + 1.2256d_1)r_2 + (1.1392 + 4.1209d_1)r_3 \\
&\qquad\qquad + (2.4554 - .3407d_1) = 0 \\
&(3.215 + 7.2675d_1)r_1 + (-1.0716 + 7.5778d_1)r_2 + (7.2675 + 6.4279d_1)r_3 \\
&\qquad\qquad + (7.2675 - 3.5721d_1) = 0 \\
&(5.922 + 8.9149d_1)r_1 + (6.7336 + 11.4763d_1)r_2 + (6.1668 + 4.2833d_1)r_3 \\
&\qquad\qquad + (8.9149 - 6.5798d_1) = 0 \\
&(1.6915 + .6489d_1)r_1 + (6.4469 + 2.5476d_1)r_2 + (-5.9209 - 2.2287d_1)r_3 \\
&\qquad\qquad + (.6489 - 1.8794d_1) = 0
\end{aligned} \tag{8.3}$$

From the above system of equations, the eliminant polynomial in d_1 is obtained by setting the determinant of the coefficients equal to zero. The eliminant polynomial is determined from equation (5.7) as

$$P(d_1) = 61.4435 - 31.5855d_1 + .7805d_1^{\,2} \tag{8.4}$$

The two real roots of $P(d_1)$ are

$$\begin{aligned}
\text{A:}\quad d_1 &= 2.0491 \\
\text{B:}\quad d_1 &= 38.4338
\end{aligned} \tag{8.5}$$

These two roots of d_1 are then substituted back into any three equations of (8.3) to get corresponding values for r_1, r_2 and r_3 (coordinates of vector $\underset{\sim}{R}$). Thus, the following two R-Sp-R mechanisms described by the design vectors $\underset{\sim}{A}$ and $\underset{\sim}{R}$ are obtained.

$$\begin{aligned} \text{A:} \quad \underset{\sim}{A} &= .4386\hat{i} + .8987\hat{j} + 0\hat{k} \\ \underset{\sim}{R} &= -.6875\hat{i} + .6127\hat{j} + .1782\hat{k} \end{aligned} \tag{8.6}$$

and

$$\begin{aligned} \text{B:} \quad \underset{\sim}{A} &= .026\hat{i} + .9993\hat{j} + 0\hat{k} \\ \underset{\sim}{R} &= 3.9642\hat{i} - 1.8686\hat{j} - 1.7588\hat{k} \end{aligned} \tag{8.7}$$

These two R-Sp-R mechanisms A and B, without the contour of the plane and the required sphere radius, are shown together with their displacement, velocity and acceleration curves in figures 8.1 and 8.2. The analysis results for mechanism A are also tabulated in table 8-1 to be used in the determination of the plane contour, the radius of the sphere and the orientation of the sphere-rod.

The methods shown in sections 7.1.1 and 7.1.2 are used to determine the plane contour, the orientation of the sphere-rod and the radius of the sphere for mechanism A. An in-plane coordinate system is defined (figure 8.3) by letting

$$\underset{\sim}{U} = 0\hat{i} + 0\hat{j} + 1\hat{k} \tag{8.8}$$

and

$$\underset{\sim}{V} = \underset{\sim}{A} \times \underset{\sim}{U} = .8987\hat{i} - .4386\hat{j} + 0\hat{k} \tag{8.9}$$

The vector $\underset{\sim}{B}$, that defines the sphere-rod orientation, is arbitrarily made to coincide initially with $\underset{\sim}{V}$.

Using the equations of sections 7.1.1 and 7.1.2, table 8-2 is constructed to show the scalars λ_{uj}, λ_{vj}, λ'_{aj}, λ'_{uj}, λ'_{vj}, $\overline{R_jP_j}$ and r_j, defined in these equations. These scalars are also shown for a (jth) position in figure 8.4.

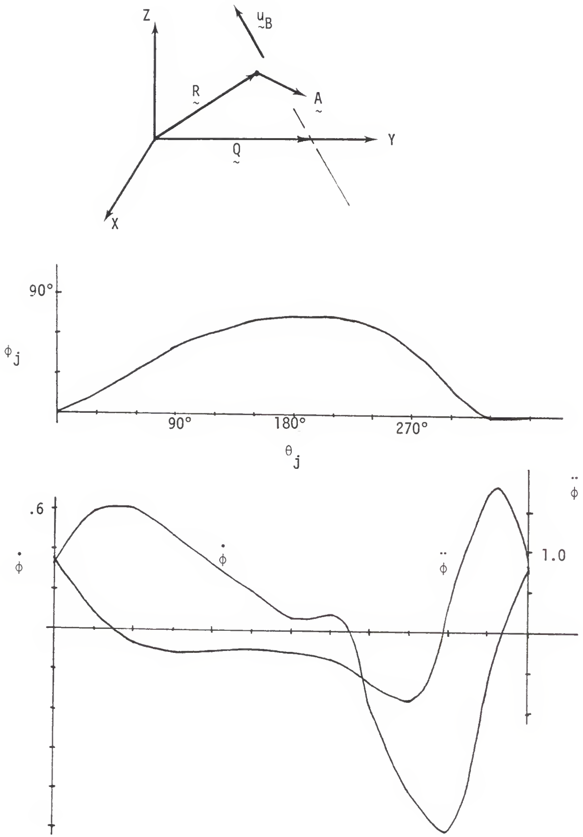


Figure 8.1 Analysis results for R-Sp-R mechanism A

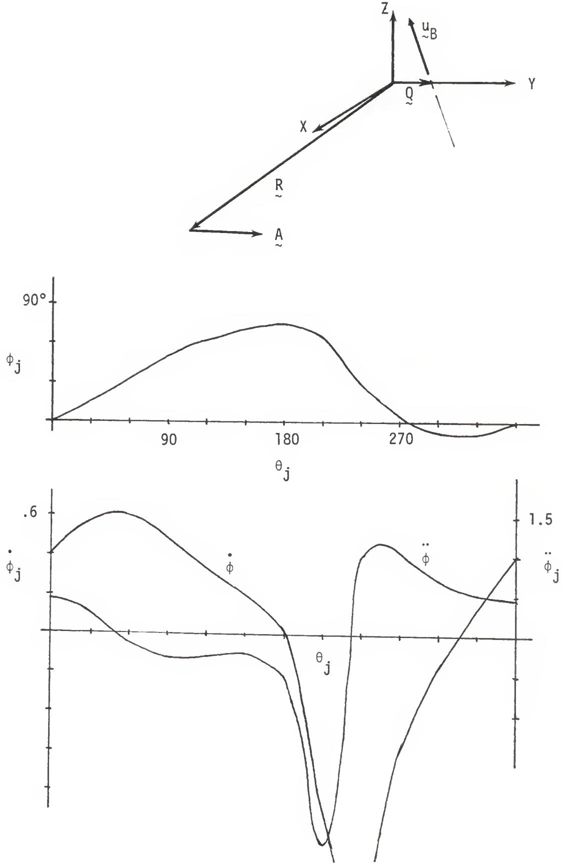


Figure 8.2 Analysis results for R-Sp-R mechanism B

TABLE 8-1

ANALYSIS OF THE JUST SYNTHESIZED R-Sp-R MECHANISM

The Mechanism Parameters are:

$$\underset{\sim}{u}_B = .3162\hat{i} + 0\hat{j} + .9487\hat{k}$$

$$\underset{\sim}{A} = .4386\hat{i} + .8987\hat{j} + 0\hat{k}$$

$$\underset{\sim}{R} = -.6875\hat{i} + .6127\hat{j} + .1782\hat{k}$$

$$\dot{\theta} = 1, \qquad \ddot{\theta} = 0$$

Input Angle θ	Output Angle ϕ	Output Velocity $\dot{\phi}$	Output Acceleration $\ddot{\phi}$
0	$\boxed{0}$	.337	.863
30	$\boxed{15}$	.607	.213
60	33.67	.604	-.173
90	$\boxed{50}$	.475	-.282
120	62.07	.331	-.263
150	$\boxed{70}$	$\boxed{.200}$	-.241
180	74.07	.069	-.269
210	73.81	.098	-.389
240	67.24	-.364	-.654
270	50.36	-.783	-.874
300	21.68	-1.025	.335
330	-.95	-.300	1.729

Note: Framed quantities are the prescribed precision points.

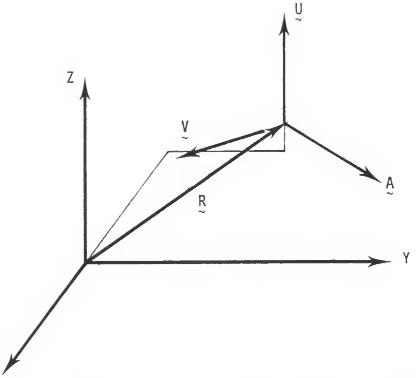


Figure 8.3 The in-plane coordinate system for R-Sp-R mechanism A

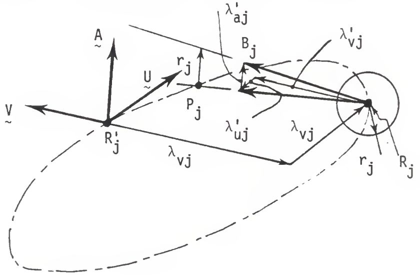


Figure 8.4 (Jth) position of $\underset{\sim}{B}_j$

TABLE 8-2

SCALARS REQUIRED FOR Sp PAIR ELEMENT DESIGN

Unit vectors of the in-plane coordinate system are

$$\underset{\sim}{A} = .4386\hat{i} + .8987\hat{j} + 0\hat{k}$$

$$\underset{\sim}{U} = 0\hat{i} + 0\hat{j} + 1\hat{k}$$

$$\underset{\sim}{V} = .8987\hat{i} - .4386\hat{j} + 0\hat{k}$$

Assumed initial sphere-rod orientation

$$\underset{\sim}{B} = \underset{\sim}{V} = .8987\hat{i} - .4386\hat{j} + 0\hat{k}$$

Position θ	λ_{uj}	λ_{vj}	λ'_{aj}	λ'_{uj}	λ'_{vj}	$\overline{R_jP_j}$	r_j
30° *	.3196	- .1239		-.1785	.9626		
60° *	.5453	- .3461		-.2898	.8815		
90°	.6243	- .6338	-.4356	-.3266	.8388	.425	.1666
100°	.5382	- .9180	-.4203	-.2775	.8638	.865	.3176
150°	.3093	-1.1204	-.3716	-.1403	.9178	1.100	.3795
180°	-.0020	-1.1804	-.3164	.0800	.9467	1.150	.3457
210°	-.3166	-1.0758	-.2699	.2825	.9205	1.050	.2729
240°	-.5504	- .8262	-.2393	.4736	.8562	.950	.2224
270°	-.6433	- .4928	-.2075	.5203	.8284	.550	.1237
300° *	-.5681	- .1721		.5932	.9025		
330° *	-.3283	- .0128		.2213	.9745		

*Point P_j does not exist because the sphere-rod does not overlap the plane element.

Figure 8.5 shows the contour of the plane element of the Sp pair as defined by λ_{uj} and λ_{vj}. The values of $\overline{R_jP_j}$ are determined graphically from figure 8.6 which shows the projection of the (jth) position of the sphere-rod vector on the plane.

Since all the λ'_{aj}-s have the same sign, the minimum required sphere radius of .3795 at $\theta = 150°$ can be reduced in half by rotating the vector $\underset{\sim}{B}$ by an angle $(\theta_a/2)$ about $\underset{\sim}{U}$ as shown in figures 8.7 and 8.8. Thus, the final orientation of the sphere-rod is defined by

$$\underset{\sim}{B} = .9588\hat{i} - .2843\hat{j} + 0\hat{k} \qquad (8.10)$$

Assuming that a sphere-rod radius of .05 is going to be used, the final minimum required radius of the sphere r' is (see figure 8.9)

$$r' = .3795/2 + .05\csc 9.5° = .2405 \qquad (8.11)$$

The resulting R-Sp-R mechanism is shown in figure 8.10.

8.2 Six-FSP Synthesis of a Revolute--Sphere-Plane--Revolute Mechanism

An R-Sp-R mechanism is to be synthesized to satisfy the following six prescribed finitely separated positions.

At $j = 1, \quad \theta_1 = 0° \quad \phi_1 = 0°$
$j = 2, \quad \theta_2 = 15° \quad \phi_2 = 7°$
$j = 3, \quad \theta_3 = 40° \quad \phi_3 = 21°$
$j = 4, \quad \theta_4 = 70° \quad \phi_4 = 39°$
$j = 5, \quad \theta_5 = 100° \quad \phi_5 = 54°$
$j = 6, \quad \theta_6 = 130° \quad \phi_6 = 65°$

Using the K_{ij}-s of Table 8-3 and equation (5.9) will give the following cubic eliminant:

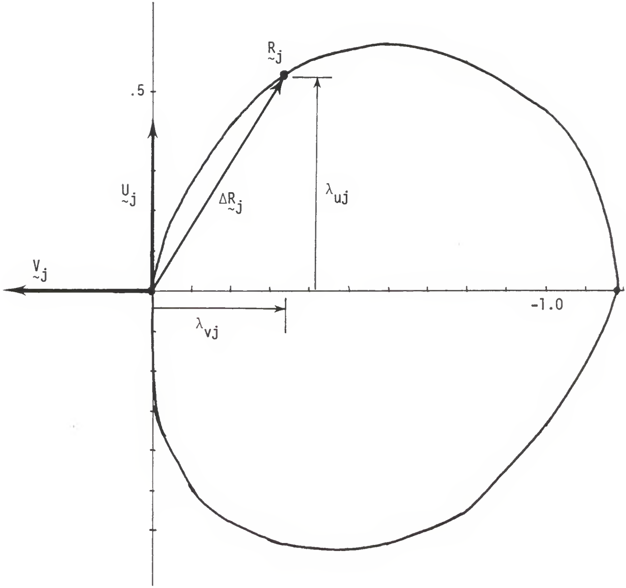


Figure 8.5 Plane contour defined by λ_{uj} and λ_{vj}

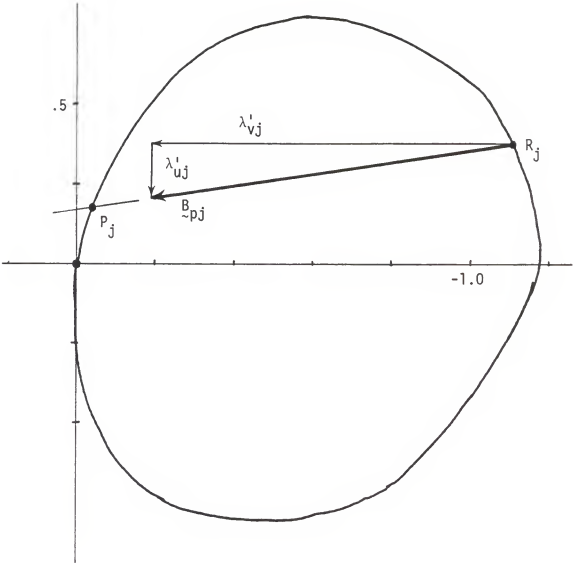


Figure 8.6 (Jth) position of $\underset{\sim}{B}_{pj}$

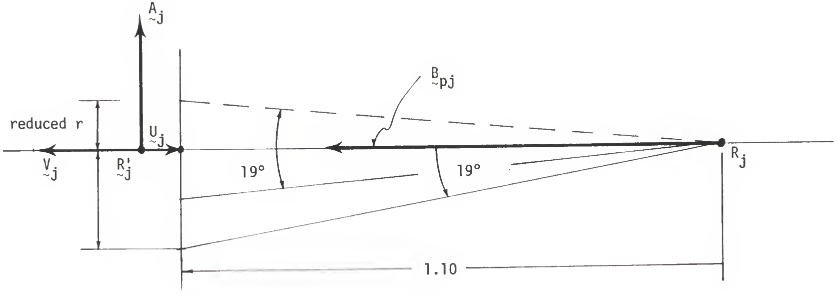


Figure 8.7 Reduction of the sphere-radius

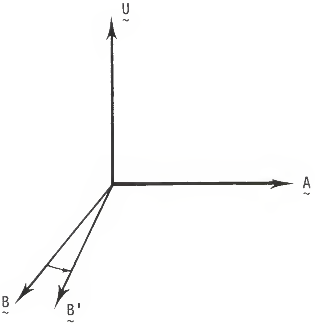


Figure 8.8 Rotation of $\underset{\sim}{B}$ about $\underset{\sim}{U}$ towards $\underset{\sim}{A}$

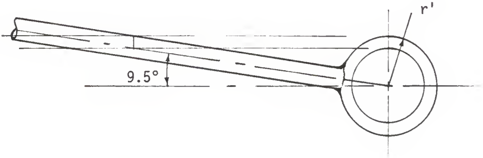


Figure 8.9 Sphere-radius r' including the effect of the radius of the sphere-rod

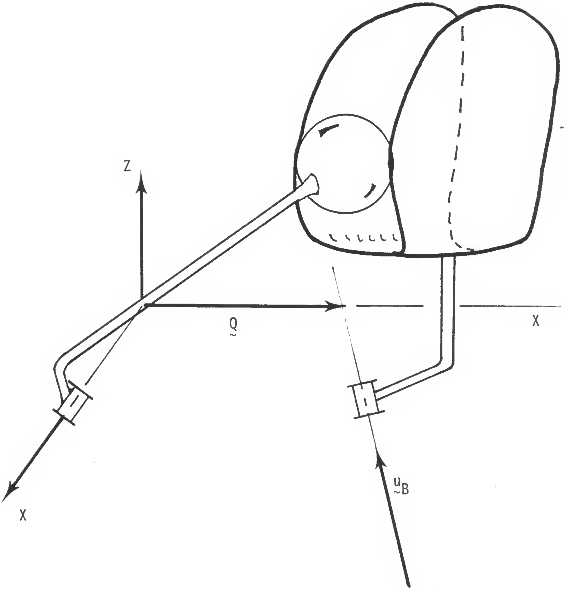


Figure 8.10 The synthesized R-Sp-R mechanism

TABLE 8-3

THE K_{ij}-s FOR A SIX-FSP SYNTHESIS OF AN R-Sp-R

$$\underset{\sim}{u}_B = .3162\hat{i} + 0\hat{j} + .9487\hat{k}$$

	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
	$\theta_2 = 15°$	$\theta_3 = 40°$	$\theta_4 = 70°$	$\theta_5 = 100°$	$\theta_6 = 130°$
	$\phi_2 = 7°$	$\phi_3 = 21°$	$\phi_4 = 39°$	$\phi_5 = 54°$	$\phi_6 = 65°$
K_{1j}	-.0067	-.0598	-.2006	-.3710	- .5197
K_{2j}	-.1156	-.3400	-.5970	-.7675	- .8598
K_{3j}	.1123	.2732	.2670	-.0115	- .4200
K_{4j}	-.0313	-.2120	-.5472	-.8501	-1.0521
K_{5j}	.2214	.5517	.8507	.9887	.9060
K_{6j}	-.0278	-.2033	-.5382	-.7773	-.7700
K_{7j}	-.2197	-.5133	-.6622	-.6233	-.5080
K_{8j}	-.0248	-.1662	-.4786	-.9146	-1.3862

The R-Sp-R Synthesis Equation is

$$[(1 + pd_2)K_{1j} + K_{2j}d_1]r_1 + [K_{3j} + K_{4j}d_1 + K_{5j}d_2]r_2$$

$$+ [K_{6j} + K_{7j}d_1 + K_{8j}d_2]r_3 + [(1 + pd_2)K_{2j} + qK_{1j}d_1] = 0$$

$$j = 2,3,4,5,6$$

where $p = -b_1/b_3 = -1/3$ and $q = -1/b_3^2 = -1/.9$

$$P(d_2) = .2061 - .7717d_2 + .6613d_2^{\,2} - .8367d_2^{\,3} = 0 \qquad (8.13)$$

The only real root of $P(d_2)$ is $d_2 = .319$. The corresponding value of d_1 is obtained by using equation (5.11), giving

$$P(d_1) = .021054 + .0084d_1 = 0 \qquad (8.14)$$

The value of $d_1 = -2.506$ and $d_2 = .319$ are then substituted into any three of the five synthesis equations. Using the first three will give the following system of equations.

$$\begin{aligned} .2837r_1 + .2614r_2 + .5149r_3 &= .122 \\ .7986r_1 + .9805r_2 + 1.03r_3 &= .4716 \\ 1.3161r_1 + 1.9097r_2 + .9686r_3 &= 1.0942 \end{aligned} \qquad (8.15)$$

Solving equation (8.15) will give $r_1 = 2.4056$, $r_2 = -.7175$ and $r_3 = -.7242$. Thus, the following R-Sp-R mechanism defined by $\underset{\sim}{R}$ and $\underset{\sim}{A}$ is obtained.

$$\underset{\sim}{R} = 2.4056\hat{i} - .7175\hat{j} - .7242\hat{k} \qquad (8.16)$$

$$\underset{\sim}{A} = .3681\hat{i} - .9225\hat{j} + .1174\hat{k} \qquad (8.17)$$

8.3 Three-FSP Synthesis of a Revolute--Cylinder-Plane--Cylinder Mechanism

An R-Cp-C mechanism is to be synthesized to satisfy the following finitely separated positions:

$$\begin{aligned} \text{At}\quad j &= 1, & \theta_1 &= 0 & \phi_1 &= 0 & S_{c1} &= 0 \\ j &= 2, & \theta_2 &= 60° & \phi_2 &= 50° & S_{c2} &= 1 \\ j &= 3, & \theta_3 &= 100° & \phi_3 &= 90° & S_{c3} &= 3 \end{aligned} \qquad (8.18)$$

In addition, the axis of the cylinder joint is given as

$$\underset{\sim}{u}_B = .7071\hat{i} + 0\hat{j} + .7071\hat{k} \qquad (8.19)$$

The synthesis case of C3-1 is chosen wherein the unknowns are d_1, d_2, e_1, r_2 and r_3 (see figure 3.6 and equation (4.32)). The other mechanism parameters are arbitrarily assigned as

$$\begin{aligned} e_2 &= .5 \\ \text{and} \qquad r_1 &= 3.0 \end{aligned} \tag{8.20}$$

From the required θ_j-s, ϕ_j-s, S_{cj}-s and the given axis $\underset{\sim}{u}_B$, the L_{ij} and M_{ij} constants are evaluated and substituted in the R-Cp-C synthesis equations (5.12 to 5.14) to give the following two systems of equations.

$$e_1d_1 + .5d_2 + 1 = 0$$

$$(-6.846 + 7.905e_1)d_1 + (6.185 + 4.405e_1)d_2 + (6.315 + 4.255e_1) = 0$$

$$(-7.685 + 6.964e_1)d_1 + (8.048 + 6.125e_1)d_2 + (1.084 + 3.696e_1) = 0 \tag{8.21}$$

and

$$(-1.786 - 5.417d_1 + 1.786d_2)3 + (4.255 - 2.095d_1 + 4.405d_2)r_2$$
$$+ (-3.798 - 2.858d_1 - 1.202d_2)r_3 + (-12.488 + 3.572d_1 - 1.654d_2) = 0$$

$$(-5 - 7.071d_1 + 5d_2)3 + (3.696 - 3.036d_1 + 6.152d_2)r_2$$
$$+ (-7.832 - 1.228d_1 - 3.905d_2)r_3 + (-28.784 + 10d_1 - 14.12d_2) = 0 \tag{8.22}$$

The systems of equations (8.21) and (8.22) are solved by first solving for e_1, d_1 and d_2 from system (8.21) and then substituting the values for d_1 and d_2 into (8.22) to get r_2 and r_3.

From equation (8.21), the eliminant polynomial in e_1 is obtained as

$$P(e_1) = -28.12 - 17.645e_1 - 27.34e_1^2 - 9.781e_1^3 = 0 \tag{8.23}$$

The only one real root of $P(e_1)$, (e_1 = -2.5312), is substituted back into any two equations of (8.21) to yield d_1 = .1053 and d_2 = -1.4669. These values of d_1 and d_2 are then substituted into the two equations of (8.22) to give r_2 = -6.6785 and r_3 = -3.5981. Thus, the mechanism vectors are obtained as

$$\begin{aligned} \underset{\sim}{A} &= .5623\hat{i} + .0592\hat{j} - .8248\hat{k} \\ \underset{\sim}{C} &= .3614\hat{i} - .9148\hat{j} + .1807\hat{k} \qquad (8.24) \\ \text{and} \quad \underset{\sim}{R} &= 3\hat{i} - 6.6785\hat{j} - 3.5981\hat{k} \end{aligned}$$

Figure 8.11 shows the curves of ϕ_j and S_{cj} with respect to the input angle θ_j in the range of θ_j from 0° to 100°.

The vector $\underset{\sim}{B}$ that defines the extent of the plane element is made perpendicular to $\underset{\sim}{A}$ and $\underset{\sim}{C}$. Thus,

$$\underset{\sim}{B} = \underset{\sim}{A} \times \underset{\sim}{C} = -.7438\hat{i} - .3997\hat{j} - .5358\hat{k} \qquad (8.25)$$

Using the method of section 7.2, the length of the plane in the direction of $\underset{\sim}{B}$ is 7.21 while the required length of the cylinder in the direction of $\underset{\sim}{C}$ is 4.63. The synthesized mechanism is shown in figure 8.12.

8.4 Three-FSP Synthesis of a Revolute--Sphere-Plane--Prismatic Mechanism

The R-Sp-P mechanism is to be synthesized to satisfy the following finitely separated positions:

$$\begin{aligned} \text{At} \quad j &= 1, \quad \theta_1 = 0 \quad S_{c1} = 0 \\ j &= 2, \quad \theta_2 = 30° \quad S_{c2} = .250 \qquad (8.26) \\ j &= 3, \quad \theta_3 = 45° \quad S_{c3} = .375 \end{aligned}$$

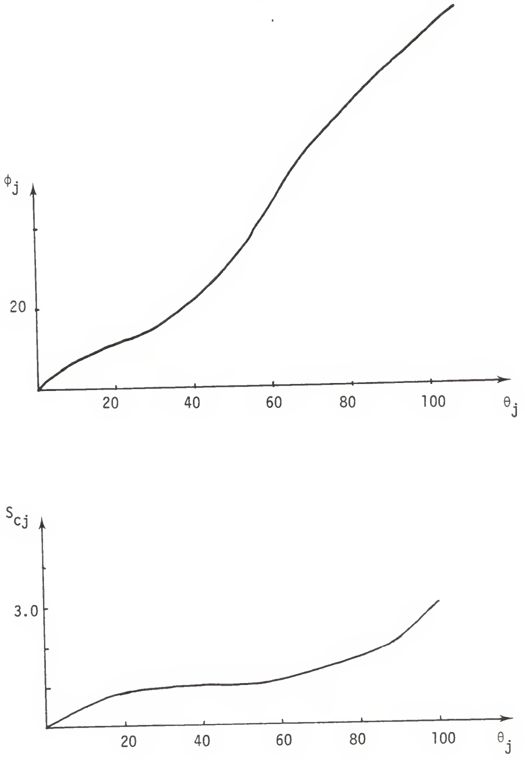


Figure 8.11 Displacement analysis results of the synthesized R-Cp-C mechanism

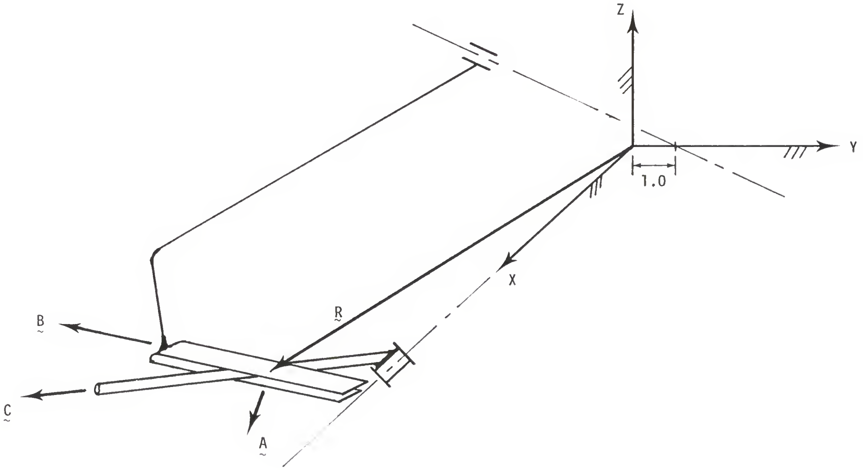


Figure 8.12 The synthesized R-Cp-C mechanism

The direction of the axis of the prismatic pair is assumed given by the vector,

$$\underset{\sim}{u}_B = .8944\hat{i} + 0\hat{j} + .4472\hat{k} \qquad (8.27)$$

Using the equations in section 4.3 will give the following system of equations:

$$-.134d_1 + .3882d_2 = .2236$$

and $\qquad (8.28)$

$$-.2929d_1 + .5394d_2 = .3354$$

This will give $d_1 = -.2316$ and $d_2 = .4961$ from which $\underset{\sim}{A}$ is obtained as $(.8771\hat{i} - .2031\hat{j} + .4351\hat{k})$. Table 8.4 shows the displacement, velocity and acceleration analysis results.

8.5 Three-FSP Synthesis of a Revolute--Sphere-Groove--Cylinder Mechanism

An R-Sg-C mechanism is to be synthesized to satisfy the following prescribed three-FSP specifications:

$$\begin{array}{llll} \text{At} \quad j = 1, & \theta_1 = 0 & \phi_1 = 0 & S_{c1} = 0 \\ j = 2, & \theta_2 = 30° & \phi_2 = 25° & S_{c2} = .65 \\ j = 3, & \theta_3 = 60° & \phi_3 = 48° & S_{c3} = .80 \end{array} \qquad (8.29)$$

The axis of the grounded cylinder pair is defined by

$$\underset{\sim}{u}_B = .9487\hat{i} + 0\hat{j} + .3162\hat{k} \qquad (8.30)$$

Letting $d_2 = -.5$ and considering r_1, r_2, r_3 and d_1 as the unknowns will give the following system of six equations in the unknowns r_1, r_2, r_3, a_1S_{g2}, a_1S_{g3} and 1:

$$\begin{bmatrix} -.0937 & -1.3363 & .2811 & (9.7658 - 1.3363d_1) & 0 & 7.5029 \\ 1.3363 & .4028 & .9906 & (3.341 + 9.0631d_1) & 0 & .9369 \\ .2811 & -.9906 & .4966 & (-4.2974 + 4.0094d_1) & 0 & -1.9541 \\ -.3308 & -2.3498 & .9925 & 0 & (9.173 - 2.3498d_1) & 9.9394 \\ 2.3498 & 1.6913 & 1.61 & 0 & (5.8749 + 6.6913d_1) & 3.3087 \\ .9925 & -1.61 & 2.0222 & 0 & (-2.5186 + 7.0502d_1) & -4.5206 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ a_1 S_{g2} \\ a_1 S_{g3} \\ 1 \end{Bmatrix} = 0 \qquad (8.31)$$

The resulting eliminant (in terms of d_1) of equation (8.31) is the quadratic

$$P(d_1) = d_1^2 - .36345d_1 - .24867 \qquad (8.32)$$

The two roots of $P(d_1)$ are .7125 and -.3490. These two roots of d_1 are then substituted into any five equations of (8.31) to get the corresponding values of r_1, r_2 and r_3. From the values of d_1, d_2, r_1, r_2, and r_3, the following two mechanisms are obtained:

$$\text{A)} \quad \begin{aligned} \underset{\sim}{A} &= .7543\hat{i} + .5374\hat{j} - .3772\hat{k} \\ \underset{\sim}{R} &= .0717\hat{i} + 1.8872\hat{j} + 5.5115\hat{k} \end{aligned} \qquad (8.33)$$

and

$$\text{B)} \quad \begin{aligned} \underset{\sim}{A} &= .8538\hat{i} - .298\hat{j} - .4269\hat{k} \\ \underset{\sim}{R} &= -3.6701\hat{i} + 2.0339\hat{j} + 3.2844\hat{k} \end{aligned} \qquad (8.34)$$

TABLE 8-4

ANALYSIS RESULTS OF THE SYNTHESIZED R-Sp-P MECHANISM

The mechanism parameters are

$$\underset{\sim}{u}_B = .8944\hat{i} + 0\hat{j} + .4472\hat{k}$$

$$\underset{\sim}{A} = .8771\hat{i} - .2031\hat{j} + .4351\hat{k}$$

$$\dot{\theta} = 1, \quad \ddot{\theta} = 0$$

Input Angle θ	Output Translation S_{cj}	Output Velocity $\dot{S}_{cj}$	Output Acceleration $\ddot{S}_{cj}$
0	0	.207	.207
15	.122	.315	.085
30	.250	.402	-.043
45	.375	.461	-.168
60	.489	.489	-.281
75	.583	.483	-.376
90	.652	.444	-.444
105	.690	.376	-.483
120	.696	.281	-.489
135	.668	.168	-.461
150	.609	.043	-.402

Tables 8-5 and 8-6 show the displacement analysis results for the two R-Sg-C mechanisms that satisfy the required motion specifications.

TABLE 8-5

DISPLACEMENT ANALYSIS OF THE SYNTHESIZED R-Sg-C MECHANISM A

The mechanism parameters are

$$\underset{\sim}{u}_B = .9487\hat{i} + 0\hat{j} + .3162\hat{k}$$

$$\underset{\sim}{A} = .7543\hat{i} + .5374\hat{j} - .3772\hat{k}$$

$$\underset{\sim}{R} = .0717\hat{i} + 1.8872\hat{j} + 5.5115\hat{k}$$

Input Angle θ	Output Angle ϕ	Groove Displacement S_g	Output Translation S_c
0	0	0	0
30	25.00	- .981	.650
60	48.00	-1.936	.800
90	70.68	-2.555	.378
120	94.69	-2.644	- .520
150	121.64	-2.181	-1.653
180	152.93	-1.314	-2.702

TABLE 8-6

DISPLACEMENT ANALYSIS OF THE SYNTHESIZED R-Sg-C MECHANISM B

The mechanism parameters are

$$\underset{\sim}{u}_B = .9487\hat{i} + 0\hat{j} + .3162\hat{k}$$

$$\underset{\sim}{A} = .8538\hat{i} - .298\hat{j} - .4269\hat{k}$$

$$\underset{\sim}{R} = -3.6701\hat{i} + 2.0339\hat{j} + 3.2844\hat{k}$$

Input Angle θ	Output Angle ϕ	Groove Displacement S_g	Output Translation S_c
0	0	0	0
30	25.00	- .693	.650
60	48.00	-1.129	.800
90	69.51	-1.213	.423
120	90.19	- .860	- .420
150	111.10	- .057	-1.578
180	134.12	1.092	-2.814

CHAPTER 9

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDY

In this dissertation, a method for the synthesis and analysis of three-link spatial function generators have been presented. The minimum number of only three-links (two moving links) was achieved by using higher pairs as the intermediate connecting joints between two moving links. Four mechanisms, R-Sp-R, R-Sp-P, R-Cp-C and R-Sg-C were considered for synthesis as function generators whereby, input rotations were coordinated with output rotations, output translations and output motions of a combination of rotations and translations.

Procedures for multiply-separated precision-point synthesis were shown for different numbers of unknowns and choices of specified and unknown parameters for each of the four mechanisms. In addition to the geometric synthesis of the mechanism, design methods for the dimensional synthesis of the higher pair elements were also shown.

The procedures and methods used are based on vector representations of geometric and motion parameters. Vector notation was utilized to simplify and shorten derivations and to obtain more compact expressions. The use of vectors, especially in the governing constraint equations, also aided in visualizing the different concepts and methods utilized in this undertaking.

The results of this work can be considered useful from two different aspects. First, the results of this work can be utilized by practicing engineers and the like to synthesize spatial function generators. Secondly, this work provides part of the groundwork for further work in the area of synthesis and analysis of spatial mechanisms with higher pairs.

The results in this dissertation can be applied by incorporating it in an interactive computer package, such as CADSPAM--Computer Aided Design of Spatial Mechanisms. Envisioned to include the analytical procedures with both numerical and graphical outputs, CADSPAM will enable a designer to "see" the resulting mechanisms as they are synthesized. By making use of the simpler synthesis cases, results can be obtained in a shorter period of time. The designer can then adjust the specifications of the prescribed parameters to achieve the desired mechanism characteristics.

As for laying part of the groundwork for further studies, the following areas related to this dissertation are recommended:

1. Synthesis of four-link spatial mechanisms with higher pairs for motion generation or path generation.
2. Optimization of the synthesis procedures by considering other properties like transmission/deviation angles or pair element dimensions as additional constraints or specifications.
3. The feasibility of synthesizing mechanisms with higher pairs that have curvatures in the pair elements such that a function specification can be achieved exactly--not only at the precision points but throughout the motion cycle.
4. Avoidance of branching and order problems in the synthesis of the three-link spatial function generators.

APPENDIX A

EVALUATION OF THE ELIMINANT POLYNOMIAL FROM DETERMINANTS

The main objective in the evaluation of the determinant is to obtain the polynomial from the determinant since the elements are functions of a variable.

The following are properties, methods or techniques that aid in the evaluation of the determinants.

The notation,

$$\begin{bmatrix} (A_j + B_jx) & (C_j + D_jx) & (E_j + F_jx) & \cdots\cdots \\ 0 & (J_j + K_jx) & (L_j + M_jx) & \cdots\cdots \end{bmatrix} \tag{A.1}$$

$$j = 1, 2 \ldots n$$

is introduced for the matrix

$$\begin{bmatrix} (A_1 + B_1x) & (C_1 + D_1x) & (E_1 + F_1x) & \cdots\cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ (A_n + B_nx) & (C_n + D_nx) & (E_n + F_nx) & \cdots\cdots \\ 0 & (J_1 + K_1x) & (L_1 + M_1x) & \cdots\cdots \\ 0 & \cdot & \cdot & \cdots\cdots \\ \cdot & \cdot & \cdot & \cdots\cdots \\ \cdot & (J_n + K_nx) & (L_n + M_nx) & \cdots\cdots \end{bmatrix} \tag{A.2}$$

A. If each of the elements in a row or column consists of two terms, then the determinant can be expressed as the sum of two terms such that:

$$|(A_j + B_j) \quad C_j \quad D_j| = |A_j \quad C_j \quad D_j| + |B_j \quad C_j \quad D_j| \tag{A.3}$$

B. Multiplying each element of one row or one column is equivalent to multiplying the determinant by the same quantity.

$$|aA_j \quad B_j \quad C_j| = a\,|A_j \quad B_j \quad C_j| \tag{A.4}$$

Properties A and B are useful for evaluating a determinant since these properties can be extended to apply to additional rows or columns. Using these two properties, an eliminant polynomial can be obtained without carrying out any algebraic manipulation of the variable in the elements. Applied to a simple example,

$$\begin{aligned} |(A_j + B_jx) \quad (C_j + D_jx)| &= |A_j \quad (C_j + D_jx)| + |B_jx \quad (C_j + D_jx)| \\ &= |A_j \quad C_j| + (|A_j \quad D_j| + |B_j \quad C_j|)x \\ &\quad + |B_j \quad D_j|x^2 \end{aligned} \tag{A.5}$$

C. Just as the elements of a row or column and their corresponding cofactors can be used to expand a determinant into lower order determinants, so can expansion be carried out about two or more rows or columns. This method of determinant expansion is called Laplace's Development. The sign of the cofactor expansion about the $k_a.....k_b$ rows and the $l_{a'}, ...l_{b'}$ columns is:

$$-1^{(k_a + ..k_b + l_{a'} + ..l_{b'})} \tag{A.6}$$

where: within a matrix,

$$\begin{bmatrix} a_{k_a l_a}, & \cdots\cdots\cdots & a_{k_a l_b}, \\ \vdots & & \\ a_{k_b l_a}, & \cdots\cdots\cdots & a_{k_b l_b}, \end{bmatrix} \qquad (A.7)$$

Property C is useful in evaluating the determinant of the matrix and also when we are only interested in obtaining the degree of the resulting polynomial.

APPENDIX B

THE K_{ij}^n COEFFICIENTS OF THE R-Sp-R SYNTHESIS EQUATION FOR n = 0, n = 1 and n = 2

Let: $E_j = \sin\phi_j \cos\theta_j$

$F_j = \cos\phi_j \sin\theta_j$

$G_j = \sin\phi_j \sin\theta_j$

$H_j = \cos\phi_j \cos\theta_j$

and $b_1 = \cos\theta_s$

$b_3 = -\sin\theta_s$

where: θ_s is the skew angle between the axes $\underset{\sim}{u}_A$ and $\underset{\sim}{u}_B$ (see figure 3.5)

$$K_{1j} = b_3^2(\cos\phi_j - 1)$$

$$K_{1j}^1 = -b_3^2 \sin\phi_j \dot{\phi}_j$$

$$K_{1j}^2 = -b_3^2(\cos\phi_j \dot{\phi}_j^2 + \sin\phi_j \ddot{\phi}_j)$$

$$K_{2j} = -b_3\sin\phi_j$$

$$K^1_{2j} = -b_3\cos\phi_j\dot{\phi}_j$$

$$K^2_{2j} = b_3(\sin\phi_j\dot{\phi}_j^2 - \cos\phi_j\ddot{\phi}_j)$$

$$K_{3j} = b_1b_3\sin\theta_j + A_{3j}$$

$$K^1_{3j} = B_{3j}\dot{\phi}_j + C_{3j}\dot{\theta}_j$$

$$K^2_{3j} = -A_{3j}\dot{\phi}_j^2 + (-b_1b_3\sin\theta_j - A_{3j})\dot{\theta}_j^2 + 2(-b_3F_j + b_1b_3E_j)\dot{\phi}_j\dot{\theta}_j$$

$$+ B_{3j}\ddot{\phi}_j + C_{3j}\ddot{\theta}_j$$

where: $A_{3j} = b_3E_j - b_1b_3F_j$

$B_{3j} = b_3H_j + b_1b_3G_j$

$C_{3j} = b_1b_3\cos\theta_j - b_3G_j - b_1b_3H_j$

$$K_{4j} = -1 + A_4j$$

$$K^1_{4j} = B_{4j}\dot{\phi}_j + C_{4j}\dot{\theta}_j$$

$$K^2_{4j} = -A_{4j}(\dot{\phi}_j^2 + \dot{\theta}_j^2) + 2(b_1H_j + G_j)\dot{\phi}_j\dot{\theta}_j + B_{4j}\ddot{\phi}_j + C_{4j}\ddot{\theta}_j$$

where: $A_{4j} = b_1G_j + H_j$

$B_{4j} = b_1F_j - E_j$

$C_{4j} = b_1E_j - F_j$

$$K_{5j} = b_3^2\sin\theta_j + A_{5j}$$

$$K_{5j}^1 = B_{5j}\dot{\phi}_j + C_{5j}\dot{\theta}_j$$

$$K_{5j}^2 = -A_{5j}\dot{\phi}_j^2 + (-b_3^2\sin\theta_j - A_{5j})\dot{\theta}_j^2 + 2(b_1F_j - b_1^2E_j)\dot{\phi}_j\dot{\theta}_j$$

$$+ B_{5j}\ddot{\phi}_j + C_{5j}\ddot{\theta}_j$$

where: $A_{5j} = -b_1E_j + b_1^2F_j$

$$B_{5j} = -b_1(H_j + b_1G_j)$$

$$C_{5j} = b_3^2\cos\theta_j + b_1G_j + b_1^2H_j$$

$$K_{6j} = b_1b_3\cos\theta_j + A_{6j}$$

$$K_{6j}^1 = B_{6j}\dot{\phi}_j + C_{6j}\dot{\theta}_j$$

$$K_{6j}^2 = -A_{6j}\dot{\phi}_j^2 + (-b_1b_3\cos\theta_j - A_{6j})\dot{\theta}_j^2 - 2(b_3H_j + b_1b_3G_j)\dot{\phi}_j\dot{\theta}_j$$

$$+ B_{6j}\ddot{\phi}_j + C_{6j}\ddot{\theta}_j$$

where: $A_{6j} = -b_3G_j - b_1b_3H_j$

$$B_{6j} = -b_3F_j + b_1b_3E_j$$

$$C_{6j} = -b_1b_3\sin\theta_j - b_3E_j + b_1b_3F_j$$

$$K_{7j} = A_{7j}$$

$$K_{7j}^1 = B_{8j}\dot{\phi}_j + C_{8j}\dot{\theta}_j$$

$$K_{7j}^2 = -A_{7j}(\dot{\phi}_j^2 + \dot{\theta}_j^2) + 2(E_j - b_1F_j)\dot{\phi}_j\dot{\theta}_j + B_{7j}\ddot{\phi}_j + C_{7j}\ddot{\theta}_j$$

$$A_{7j} = b_1E_j - F_j$$

$$B_{7j} = b_1H_j + G_j$$

$$C_{7j} = -b_1G_j - H_j$$

$$K_{8j} = -1 + b_3^2\cos\theta_j + A_{8j}$$

$$K_{8j}^1 = B_{8j}\dot{\phi}_j + C_{8j}\dot{\theta}_j$$

$$K_{8j}^2 = -A_{8j}\dot{\phi}_j^2 + (-b_3^2\cos\theta_j - A_{8j})\dot{\theta}_j^2 + 2(b_1H_j + b_1^2G_j)\dot{\phi}_j\dot{\theta}_j$$

$$+ B_{8j}\ddot{\phi}_j + C_{8j}\ddot{\theta}_j$$

where: $A_{8j} = b_1G_j + b_1^2H_j$

$$B_{8j} = b_1F_j - b_1^2E_j$$

$$C_{8j} = -b_3^2\sin\theta_j + b_1E_j - b_1^2F_j$$

APPENDIX C

THE L_{ij}^{n} and M_{ij}^{n} COEFFICIENTS OF THE R-CP-C SYNTHESIS EQUATION FOR n = 0, n = 1 AND n = 2

The K_{ij}^{n}-s referred to in this appendix are those of the R-Sp-R synthesis equation.

$L_{1j} = 1 + K_{1j}$

$L_{1j}^{n} = K_{1j}^{1}$

$L_{1j}^{2} = K_{1j}^{2}$

$L_{2j}^{n} = K_{3j}^{n}$

$L_{3j}^{n} = K_{6j}^{n}$

$L_{4j}^{n} = K_{2j}^{n}$

$L_{5j} = 1 + K_{4j}$

$L_{5j}^{2} = K_{4j}^{2}$

$L_{5j}^{3} = K_{4j}^{3}$

$L_{6j}^{n} = K_{7j}^{n}$

$$L_{7j}^{n} = pK_{1j}^{n}$$

$$L_{8j}^{n} = K_{5j}^{n}$$

$$L_{9j} = 1 + K_{8j}$$

$$L_{9j}^{1} = K_{8j}^{1}$$

$$L_{9j}^{2} = K_{8j}^{n}$$

$$M_{ij}^{n} = K_{ij}^{n} \quad \text{for } i = 1, 2, 3 \ldots.8 \text{ and } n = 0, 1, 2$$

$$M_{9j}^{n} = K_{2j}^{n} - b_1 S_{cj}^{n}$$

$$M_{10j}^{n} = pK_{2j}^{n} - b_3 S_{cj}^{n}$$

where: $p = -b_1/b_3$

$q = -1/b_3^{2}$

$S_{cj}^{1} = \dot{S}_{cj}$

$S_{cj}^{2} = \ddot{S}_{cj}$

APPENDIX D

THE ELEMENTS OF $[G_{ikj}^{n}]$, $[H_{ikj}^{n}]$ and $\{P_{ij}\}$ OF THE R-Sg-C SYNTHESIS EQUATION

$$G_{11j} = b_3^{2}(\cos\phi_j - 1)$$

$$G_{11j}^{1} = -b_3^{2}\sin\phi_j\dot{\phi}_j$$

$$G_{11j}^{2} = -b_3^{2}(\cos\phi_j\dot{\phi}_j^{2} + \sin\phi_j\ddot{\phi}_j)$$

$$G_{12j} = -b_3\sin\phi_j$$

$$G_{12j}^{1} = -b_3\cos\phi_j\dot{\phi}_j$$

$$G_{12j}^{2} = b_3(\sin\phi_j\dot{\phi}_j^{2} - \cos\phi_j\ddot{\phi}_j)$$

$$G_{13j}^{n} = -b_1/b_3G_{11j}^{n}$$

$$G_{21j}^{n} = -G_{12j}^{n}$$

$$G_{22j} = \cos\phi_j - \cos\theta_j$$

$$G_{22j}^{1} = -\sin\phi_j\dot{\phi}_j + \sin\theta_j\dot{\theta}_j$$

$$G_{22j}^{2} = -(\cos\phi_j\dot{\phi}_j^{2} + \sin\phi_j\ddot{\phi}_j) + (\cos\theta_j\dot{\theta}_j^{2} + \sin\theta_j\ddot{\theta}_j)$$

$$G_{23j} = -b_1 \sin\phi_j + \sin\theta_j$$

$$G_{23j}^{1} = -b_1 \cos\phi_j \dot{\phi}_j + \cos\theta_j \dot{\theta}_j$$

$$G_{23j}^{2} = b_1(\sin\phi_j \dot{\phi}_j^{2} - \cos\phi_j \ddot{\phi}_j) - (\sin\theta_j \dot{\theta}_j^{2} - \cos\theta_j \ddot{\theta}_j)$$

$$G_{31j}^{n} = G_{13j}^{n}$$

$$G_{32j}^{n} = -G_{23j}^{n}$$

$$G_{33j} = b_3^{2} + b_1^{2} \cos\phi_j - \cos\theta_j$$

$$G_{33j}^{1} = -b_1^{2} \sin\phi_j \dot{\phi}_j + \sin\theta_j \dot{\theta}_j$$

$$G_{33j}^{2} = -b_1^{2}(\cos\phi_j \dot{\phi}_j^{2} + \sin\phi_j \ddot{\phi}_j) + (\cos\theta_j \dot{\theta}_j^{2} + \sin\theta_j \ddot{\theta}_j)$$

$$H_{11j} = b_1^{2} + b_3^{2} \cos\phi_j$$

$$H_{11j}^{1} = -b_3^{2} \sin\phi_j \dot{\phi}_j$$

$$H_{11j}^{2} = -b_3^{2}(\cos\phi_j \dot{\phi}_j^{2} + \sin\phi_j \ddot{\phi}_j)$$

$$H_{12j}^{n} = G_{12j}^{n}$$

$$H_{13j}^{n} = G_{13j}^{n}$$

$$H_{21j}^{n} = H_{12j}^{n}$$

$$H_{22j} = \cos\phi_j$$

$$H^1_{22j} = -\sin\phi_j\dot{\phi}_j$$

$$H^2_{22j} = -(\cos\phi_j\dot{\phi}_j{}^2 + \sin\phi_j\ddot{\phi}_j)$$

$$H_{23j} = -b_1\sin\phi_j$$

$$H^1_{23j} = -b_1\cos\phi_j\dot{\phi}_j$$

$$H^2_{23j} = b_1(\sin\phi_j\dot{\phi}_j{}^2 - \cos\phi_j\ddot{\phi}_j)$$

$$H^n_{31j} = H^n_{13j}$$

$$H^n_{32j} = -H^n_{23j}$$

$$H_{33j} = b_3{}^2 + b_1{}^2\cos\phi_j$$

$$H^1_{33j} = -b_1{}^2\sin\phi_j\dot{\phi}_j$$

$$H^2_{33j} = -b_1{}^2(\cos\phi_j\dot{\phi}_j{}^2 + \sin\phi_j\ddot{\phi}_j)$$

$$P_{1j} = -b_3S\phi_j - b_1S_{cj}$$

$$P^1_{1j} = -b_3\cos\dot{\phi}_j\phi_j - b_1\dot{S}_{cj}$$

$$P^2_{1j} = b_3(\sin\phi_j\dot{\phi}_j{}^2 - \cos\phi_j\ddot{\phi}_j) - b_1\ddot{S}_{cj}$$

$$P_{2j} = \cos\phi_j - 1$$

$$P_{2j}^{1} = -\sin\phi_j\dot{\phi}_j$$

$$P_{2j}^{2} = -(\cos\phi_j\dot{\phi}_j + \sin\phi_j\ddot{\phi}_j)$$

$$P_{3j} = b_1\sin\phi_j - b_3S_{cj}$$

$$P_{3j}^{1} = b_1\cos\phi_j\dot{\phi}_j - b_3\dot{S}_{cj}$$

$$P_{3j}^{2} = -b_1(\sin\phi_j\dot{\phi}_j - \cos\phi_j\ddot{\phi}_j)$$

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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